

UNITARY DUAL OF p -ADIC SPLIT SO_{2n+1} AND Sp_{2n} : THE GOOD PARITY CASE (AND SLIGHTLY BEYOND)

HIRAKU ATOBE AND ALBERTO MÍNGUEZ

ABSTRACT. Let F be a p -adic field, and let G be either the split special orthogonal group $\mathrm{SO}_{2n+1}(F)$ or the symplectic group $\mathrm{Sp}_{2n}(F)$. We prove that a smooth irreducible representation of good parity of G is unitary if and only if it is of Arthur type. Combined with the algorithms of [7, 26] for detecting Arthur type representations, our result leads to an explicit algorithm for checking the unitarity of any given irreducible representation of good parity. Finally, we determine the set of unitary representations that may appear as local components of the discrete automorphic spectrum.

1. INTRODUCTION

The classification of irreducible unitary representations of classical groups over local fields stands as a central problem in the representation theory of reductive groups. This program traces its origins to I. Gel'fand's work in the 1940s and 1950s on harmonic analysis and representations of locally compact groups. For general linear groups, the classification was completed by M. Tadić [43] for p -adic groups and by D. Vogan [50] for real groups in the 1980s. For inner forms of general linear p -adic groups, the classification was achieved through the combined work of many mathematicians, see in particular [44, 42, 12, 14, 13]. In the case of classical groups over a p -adic field, only partial results are known: namely, for generic and unramified representations [31, 16, 41], and for groups of rank at most three [49], with recent progress in rank four [34]. For exceptional groups, the classification for G_2 was completed by Muić [40], while spherical, generic, Iwahori-spherical, and unipotent cases for groups of type E and F were studied by Ciubotaru [20, 21, 22] and Barbasch–Ciubotaru [15]. For recent progress in the case of real groups, see [1, 2]. The unitary dual carries a natural topology, and one of the main challenges for classical groups is the large number of isolated representations (see [48]).

Let F be a non-archimedean local field. For $\mathrm{GL}_n(F)$ and its inner forms, Tadić showed that an irreducible representation is unitary if and only if it is parabolically induced from representations belonging to the Tadić building blocks: namely, unitary Speh representations and the corresponding complementary-series representations obtained from their unramified twists. Assuming the generalized Ramanujan conjecture, the local components of discrete automorphic representations are obtained by parabolic induction from unitary Speh representations.

A simplification of Tadić's proof was achieved in [30] (see also the references therein for earlier approaches to inner forms), where one of the key steps was to avoid the use of a

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theorem of J. Bernstein [17] that played a central role in the original proof. Bernstein's method relies on special properties of the mirabolic subgroup of $\mathrm{GL}_n(F)$ that do not extend to other groups, and thus the extension to inner forms of $\mathrm{GL}_n(F)$ originally required input from type theory, trace formula methods, etc. In contrast, the approach in [30] relies only on elementary combinatorics and a careful analysis of Jacquet modules, what are now called ρ -derivatives (see Section 2.1), to control the irreducibility of the socle of parabolically induced representations.

One of the most remarkable achievements in number theory in the first quarter of the 21st century is J. Arthur's classification [3] of square-integrable automorphic representations of quasi-split symplectic and orthogonal groups. This breakthrough, accomplished through the collective effort of many mathematicians, relies on the twisted trace formula and an intricate inductive procedure, known as endoscopy. Arthur introduced what are now called *A-parameters*, which are modifications of *L-parameters* designed to capture the non-tempered behavior of local components of discrete automorphic representations, and in particular, to account for the failure of the naive Ramanujan conjecture beyond GL_N . In this paper, members in *local A-packets* are referred to as representations *of Arthur type*. They are unitary representations ([3, Theorem 2.2.1]) and they generalize the unitary Speh representations discussed above to the setting of classical groups.

Representations of Arthur type were classified by C. Mœglin [35, 36, 37, 38, 39] using purely local methods (see also [52]). Let $G = G_n$ denote either the split special orthogonal group $\mathrm{SO}_{2n+1}(F)$ or the symplectic group $\mathrm{Sp}_{2n}(F)$ of rank n , where F is a non-archimedean local field of characteristic zero. Given an *A-parameter* ψ for G , one can decompose it as

$$\psi = \psi_{\mathrm{bad}}^- \oplus \psi_{\mathrm{good}} \oplus \psi_{\mathrm{bad}}^+,$$

where ψ_{good} is a sum of irreducible self-dual representations of the same type as ψ , and $\psi_{\mathrm{bad}}^- = (\psi_{\mathrm{bad}}^+)^{\vee}$ is a sum of irreducible representations of a different type (see Section 2.6 for more details). The parameter ψ is said to be *of good parity* if $\psi_{\mathrm{bad}}^- = 0$.

There is a corresponding notion for representations. We begin by introducing some notation. If π_0 (resp. τ_i) is a smooth representation of G_{n_0} (resp. $\mathrm{GL}_{d_i}(F)$), with $d_r + \cdots + d_1 + n_0 = n$, it is customary to denote by

$$\tau_r \times \cdots \times \tau_1 \rtimes \pi_0$$

the normalized parabolically induced representation from the standard parabolic subgroup P of G_n with Levi subgroup isomorphic to $\mathrm{GL}_{d_r}(F) \times \cdots \times \mathrm{GL}_{d_1}(F) \times G_{n_0}$, induced from $\tau_r \boxtimes \cdots \boxtimes \tau_1 \boxtimes \pi_0$. An irreducible representation π of G is *of good parity* if its supercuspidal support is made out of good parity supercuspidal representations. More precisely, suppose that

$$\pi \hookrightarrow \rho_1 \times \cdots \times \rho_r \rtimes \sigma,$$

where $\rho_1, \dots, \rho_r$ and σ are supercuspidal. Then we say that π is *of good parity* if for each $1 \leq i \leq r$, there is $m_i \in \mathbb{Z}$ such that $\rho_i \cdot | \cdot |^{m_i} \rtimes \sigma$ is reducible. One can also describe this condition in terms of the *L-parameter* for π (see Section 2.6). The *A-packet* $\Pi_{\psi_{\mathrm{good}}}$ associated to ψ_{good} consists of irreducible representations of good parity.

Mœglin [35] showed that the irreducible representations in the A -packet Π_ψ are irreducible parabolic inductions of the representations in $\Pi_{\psi_{\text{good}}}$, together with the product of unitary Speh representations corresponding to ψ_{bad}^- . This reduces the classification of Arthur type representations to the case of parameters of good parity. In [36, 37, 38, 39], she ultimately classified these representations using a combination of techniques, including the analysis of reducibility in parabolic induction, the use of Jacquet functors, and what she called the *partial Aubert involution*.

Mœglin's classification was later simplified by the first-named author in the paper [5], who used ρ -derivatives to reduce the construction of representations of Arthur type of good parity to those with non-negative discrete diagonal restriction (which are easy to determine). This naturally led to the question of whether the idea in [30] could be adapted to classify unitary representations of classical groups. In this paper, we carry out this program for unitary representations of good parity of G , where G is a symplectic or split special odd orthogonal group over a non-archimedean local field F of characteristic zero. Our main theorem is as follows:

Theorem 1.1. *Let G be split $\mathrm{SO}_{2n+1}(F)$ or $\mathrm{Sp}_{2n}(F)$, and let π be an irreducible representation of G of good parity. Then π is unitary if and only if it is of Arthur type.*

In fact, we prove a stronger result that will be useful for the eventual classification of the full unitary dual. Namely, we show that the good-parity part of any irreducible unitary representation is always of Arthur type. See Corollary 4.5 for the details.

Our result proves a conjecture of Tadić, refined in [24, Conjecture 1.5], and establishes a deep connection between unitary representations and local components of discrete automorphic representations. In these works, it was also conjectured (see [48, Conjecture 1.1]) that all isolated unitary representations should be of good parity. We therefore view our theorem as a meaningful step toward the full classification of the unitary dual of classical groups. Moreover, since algorithms were developed in [7, 26] to determine whether a given irreducible representation of good parity is of Arthur type, our result shows that unitarity can be checked effectively in this case.

We now outline the main steps behind the proof of Theorem 1.1. One direction of the theorem follows from Arthur's work: every representation of Arthur type is known to be unitary ([3, Theorem 2.2.1]). This is an essential global input: for example, no other proof is known even for cotempered representations or for representations of Arthur type with discrete diagonal restriction.

For the converse, let now π be an irreducible unitary representation of $G = G_n$ of good parity. A key input we use is that for any irreducible unitary representation S of a general linear group, the parabolically induced representation

$$S \rtimes \pi$$

is also unitary and admissible, hence semisimple. By choosing S appropriately and using the explicit nature of the classification of representations of Arthur type in [5, 7], we are able to deduce that π must be of Arthur type.

Even though the underlying idea is conceptually simple, its implementation is technically involved. For any irreducible representation π of G_n of good parity, we define its *SZ-decomposition* (see Section 3): this consists of a natural sequence of irreducible representations τ_i of $\mathrm{GL}_{d_i}(F)$ (for $1 \leq i \leq r$), each induced from k_i copies of a segment-type representation, and a representation π_0 of G_{n_0} , with $d_1 + \cdots + d_r + n_0 = n$, such that if we define π_i inductively as the (irreducible) socle of $\tau_i \rtimes \pi_{i-1}$, then $\pi_r = \pi$. Hence π is a subrepresentation of the parabolically induced representation

$$\tau_r \times \cdots \times \tau_1 \rtimes \pi_0.$$

Moreover the representation π_0 has a specific constraint on its ρ -derivatives. Roughly speaking, it has almost no nonzero derivatives. We then prove the following:

Theorem 1.2 (Theorems 6.1 and 4.7). *Let π be an irreducible representation of G_n of good parity, and let $\pi_0, \pi_1, \dots, \pi_r = \pi$ be as above.*

- (1) π_0 is of Arthur type.
- (2) If π is unitary, then π_i is also of Arthur type.

As a result, Theorem 1.2 implies Theorem 1.1.

Finally, we give some details on the proof of Theorem 1.2. For part (1), if π_0 were to be of Arthur type, then the constraint on π_0 would force it to have a specific A -parameter ψ_0 . Studying this constraint using [10, Theorem 7.1], we will verify that π_0 actually lies in the packet associated with ψ_0 . See Section 6. For part (2), we show that the unitarity of π implies that for any product of copies of a unitary Speh representation S and for $1 \leq i \leq r$, the two parabolically induced representations

$$S \rtimes \pi_i \quad \text{and} \quad \mathrm{soc}(\tau_i \times S) \rtimes \pi_{i-1}$$

share a common irreducible subrepresentation. This imposes a strong constraint, which we exploit in the next step.

Given τ_i , we choose S such that the socle of $\tau_i \times S$ is a product of k_i copies of an essentially Speh representation. Let S' be a product of k_i “long” essentially Speh representations such that the socle of $S' \times \tau_i \times S$ is itself a product of k_i essentially Speh representations. We prove that the S -derivative (see Section 7.2) of

$$\mathrm{soc}(S' \times \tau_i \times S) \rtimes \pi_{i-1}$$

contains the socle of $S' \rtimes \pi_i$ (Lemma 7.5). By choosing S' appropriately, and using Mœglin’s construction, we obtain some information about the Langlands data for $\mathrm{soc}(S' \rtimes \pi_i)$. Using it together with the algorithm in [7], we deduce that π_i is of Arthur type. See Section 7.4.

In the final two sections of this article, we explore what lies beyond the good parity case. The examples of generic and unramified unitary representations suggest that the methods developed here are insufficient to fully address the general case, and that additional analytic tools will likely be necessary (see Section 9 for further discussion). However, for global applications, a particularly interesting set that extends slightly beyond the good parity case is that of representations attached to parameters which might be localizations of global A -parameters. Namely, we will consider an irreducible representation of the form

$$\pi = S_1 | \cdot |^{x_1} \times \cdots \times S_r | \cdot |^{x_r} \rtimes \pi_0,$$

where π_0 is of Arthur type of good parity, S_i is a unitary Speh representation, and $0 \leq x_i < \frac{1}{2}$ for $1 \leq i \leq r$. It is well known that not all such representations are unitary (see [24, Example 5.6 (1)]). In Theorem 8.1, we determine which of them are unitary. Notice that the same result (with the same proof) appeared in [25, Theorem 1.3] almost simultaneously.

Let us now say a few words about our restriction to symplectic and split odd orthogonal groups. We focus on these cases because the classification in [5] is currently limited to them. However, with the extension of the results in [4] to all quasi-split classical groups (see [9, Appendix C]), we expect that our methods can be adapted to this broader setting. In particular, applying a similar strategy would require extending the results of [10], [5], and [7] to other classical groups. We conjecture that all unitary representations of good parity are of Arthur type for all quasi-split classical groups. It would also be interesting to investigate whether an analogue of our result holds for real groups.

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2. PRELIMINARIES

Throughout this paper, F denotes a non-archimedean local field of characteristic zero. In this section, we recall some basic results from representation theory of p -adic groups.

2.1. ρ -Derivatives. Let n be a positive integer and fix $G \in \{\mathrm{GL}_n(F), \mathrm{SO}_{2n+1}(F), \mathrm{Sp}_{2n}(F)\}$, assumed to be split over F . We denote by $\mathrm{Rep}(G)$ the category of smooth representations of G of finite length. A representation $\Pi \in \mathrm{Rep}(G)$ is called *unitary* (or *unitarizable*) if it admits a positive-definite G -invariant Hermitian form.

We write $\mathrm{Irr}(G)$ for the set of irreducible admissible representations of G , and denote by $\mathrm{Irr}_{\mathrm{unit}}(G)$ and $\mathrm{Irr}_{\mathrm{temp}}(G)$ the subsets consisting of unitary and tempered representations, respectively. Thus, we have the chain of inclusions:

$$\mathrm{Irr}(G) \supset \mathrm{Irr}_{\mathrm{unit}}(G) \supset \mathrm{Irr}_{\mathrm{temp}}(G).$$

For $\Pi \in \mathrm{Rep}(G)$, its semisimplification is denoted by $[\Pi]$. We view this as an element of the Grothendieck group

$$\mathcal{R}(G) = \bigoplus_{\pi \in \mathrm{Irr}(G)} \mathbb{Z}[\pi],$$

and we often identify $\pi \in \mathrm{Irr}(G)$ with its image $[\pi]$ in $\mathcal{R}(G)$. For $\Pi_1, \Pi_2 \in \mathrm{Rep}(G)$, we write

$$[\Pi_1] \geq [\Pi_2]$$

if there exists $\Pi \in \mathrm{Rep}(G)$ such that $[\Pi] = [\Pi_1] - [\Pi_2]$ in $\mathcal{R}(G)$.

Given $\tau \in \mathrm{Rep}(\mathrm{GL}_d(F))$ and a character χ of $F^\times$, we denote by $\tau\chi$ the twist $\tau \otimes \chi \circ \det$.

Let P be a standard parabolic subgroup of G whose Levi subgroup is isomorphic to $\mathrm{GL}_{d_1}(F) \times \cdots \times \mathrm{GL}_{d_r}(F) \times G_0$, with G_0 of the same type as G . Let $\pi \in \mathrm{Rep}(G_0)$ and $\tau_i \in \mathrm{Rep}(\mathrm{GL}_{d_i}(F))$ for $1 \leq i \leq r$. We denote the normalized parabolically induced representation by

$$\tau_1 \times \cdots \times \tau_r \rtimes \pi = \mathrm{Ind}_P^G(\tau_1 \boxtimes \cdots \boxtimes \tau_r \boxtimes \pi).$$

When $G = \mathrm{GL}_n(F)$, this is also written as $\tau_1 \times \cdots \times \tau_r \times \pi$.

For a standard parabolic subgroup P of G , we denote the normalized Jacquet module of Π along P by $\mathrm{Jac}_P(\Pi)$. When P is clear from the context, we simply write $\mathrm{Jac}(\Pi)$.

As usual, it will be convenient to work simultaneously with all classical groups of the same type as G . Let $n \geq 0$, and let G now be either the split group $\mathrm{SO}_{2n+1}(F)$ or $\mathrm{Sp}_{2n}(F)$. Accordingly, we define

$$\mathrm{Irr}^G = \begin{cases} \bigcup_{m \geq 0} \mathrm{Irr}(\mathrm{SO}_{2m+1}(F)) & \text{if } G = \mathrm{SO}_{2n+1}(F), \\ \bigcup_{m \geq 0} \mathrm{Irr}(\mathrm{Sp}_{2m}(F)) & \text{if } G = \mathrm{Sp}_{2n}(F), \end{cases}$$

and set

$$\mathrm{Irr}^{\mathrm{GL}} = \bigcup_{m \geq 0} \mathrm{Irr}(\mathrm{GL}_m(F)).$$

We denote by $\mathrm{Irr}_{\mathrm{temp}}^G$ the subset of Irr^G consisting of irreducible tempered representations. Let $\mathcal{E}^{\mathrm{GL}}$ (resp. $\mathcal{E}^G$) denote the subset of $\mathrm{Irr}^{\mathrm{GL}}$ (resp. Irr^G) consisting of supercuspidal representations. For $\rho \in \mathcal{E}^{\mathrm{GL}}$, its *exponent* is the unique real number $e(\rho)$ such that $\rho| \cdot |^{-e(\rho)}$ is unitary.

Fix a positive integer $d \leq n$, and let $P = P_d = MN$ be the standard maximal parabolic subgroup of G whose Levi component is $M \cong \mathrm{GL}_d(F) \times G_0$, where G_0 is a classical group of the same type as G . Fix now ρ , a supercuspidal representation of $\mathrm{GL}_d(F)$. Given $\Pi \in \mathrm{Rep}(G)$, suppose that we have a decomposition

$$[\mathrm{Jac}_P(\Pi)] = \sum_{\tau \in \mathrm{Irr}(\mathrm{GL}_d(F))} [\tau] \otimes [\Pi_\tau]$$

in the Grothendieck group $\mathcal{R}(\mathrm{GL}_d(F)) \otimes \mathcal{R}(G_0)$. Then the ρ -*derivative* of Π is defined by

$$D_\rho(\Pi) = [\Pi_\rho].$$

If $n < d$, we simply set $D_\rho(\Pi) = 0$.

Moreover, for $k \geq 0$, we define the k -th ρ -*derivative* of Π by

$$D_\rho^{(k)}(\Pi) = \frac{1}{k!} \underbrace{D_\rho \circ \cdots \circ D_\rho}_{k \text{ times}}(\Pi),$$

which is well-defined as an element of an appropriate Grothendieck group. If $x - y \in \mathbb{Z}$, we set

$$D_{\rho|\cdot|^x, \dots, \rho|\cdot|^y} = \begin{cases} D_{\rho|\cdot|^y} \circ D_{\rho|\cdot|^{y-1}} \circ \cdots \circ D_{\rho|\cdot|^{x+1}} \circ D_{\rho|\cdot|^x} & \text{if } y \geq x, \\ D_{\rho|\cdot|^y} \circ D_{\rho|\cdot|^{y+1}} \circ \cdots \circ D_{\rho|\cdot|^{x-1}} \circ D_{\rho|\cdot|^x} & \text{if } y \leq x. \end{cases}$$

If $\pi \in \mathrm{Irr}(G)$ satisfies that $D_{\rho|\cdot|^x, \dots, \rho|\cdot|^y}(\pi) \neq 0$, then by Frobenius reciprocity, we can find $\pi' \leq D_{\rho|\cdot|^x, \dots, \rho|\cdot|^y}(\pi)$ such that

$$\pi \hookrightarrow \rho| \cdot |^x \times \cdots \times \rho| \cdot |^y \rtimes \pi'.$$

If $\rho, \rho' \in \mathcal{C}^{\mathrm{GL}}$ are such that $\rho \times \rho'$ is irreducible and $\pi \in \mathrm{Rep}(G)$, then, by transitivity of Jacquet functors together with the irreducibility of $\rho \times \rho'$, we have

$$D_\rho \circ D_{\rho'}(\pi) = D_{\rho'} \circ D_\rho(\pi).$$

In particular, under these circumstances,

$$D_\rho \circ D_{\rho'}(\pi) \neq 0 \iff D_{\rho'} \circ D_\rho(\pi) \neq 0.$$

2.2. Geometric Lemma. We recall the Geometric Lemma and Tadić's formula in this subsection.

Let G be either split $\mathrm{SO}_{2n+1}(F)$ or $\mathrm{Sp}_{2n}(F)$. Fix an F -rational Borel subgroup $B = TU$ of G . Let W be the Weyl group of G . For two standard parabolic subgroups $P_1 = M_1N_1$ and $P_2 = M_2N_2$, set

$$W^{M_1, M_2} = \{w \in W \mid w(M_1 \cap B)w^{-1} \subset B, w^{-1}(M_2 \cap B)w \subset B\}.$$

For any $w \in W^{M_1, M_2}$, we define a functor

$$F_w : \mathrm{Rep}(M_1) \rightarrow \mathrm{Rep}(M_2)$$

by

$$F_w = \mathrm{Ind}_{wP_1w^{-1} \cap M_2}^{M_2} \circ \mathrm{Ad}(w) \circ \mathrm{Jac}_{w^{-1}P_2w \cap M_1},$$

where $(\mathrm{Ad}(w)\pi)(m_2) = \pi(\tilde{w}^{-1}m_2\tilde{w})$ with a fixed representative $\tilde{w} \in G$ of w .

Theorem 2.1 (Geometric Lemma [18, 2.11]). *The functor $F = \mathrm{Jac}_{P_2} \circ \mathrm{Ind}_{P_1}^G : \mathrm{Rep}(M_1) \rightarrow \mathrm{Rep}(M_2)$ is glued from functors F_w , for $w \in W^{M_1, M_2}$.*

In [46], Tadić studied W^{M_1, M_2} when P_1 and P_2 are maximal, and he got the following formula after semisimplification.

Corollary 2.2 (Tadić's formula [46, Theorems 5.4, 6.5]). *Suppose that $P = MN$ is a maximal parabolic subgroup of G with $M \cong \mathrm{GL}_m(F) \times G_0$. Then, for any finite length representation of G of the form $\tau \rtimes \pi$, we have*

$$[\mathrm{Jac}_P(\tau \rtimes \pi)] = \sum_{\substack{n_1, n_2, n_3, n_4 \geq 0 \\ n_1 + n_3 + n_4 = m}} \sum_{\substack{\tau_1 \otimes \tau_2 \otimes \tau_3 \\ \tau_4 \otimes \pi_0}} (\tau_3^\vee \times \tau_1 \times \tau_4) \otimes (\tau_2 \rtimes \pi_0),$$

where

- $\tau_1 \otimes \tau_2 \otimes \tau_3$ runs over irreducible representations appearing in $[\mathrm{Jac}_R(\tau)]$ where R is the standard parabolic subgroup corresponding to the partition (n_1, n_2, n_3) ;
- $\tau_4 \otimes \pi_0$ runs over irreducible representations appearing in $[\mathrm{Jac}_{P'}(\pi)]$ where P' is the standard parabolic subgroup whose Levi is of the form $\mathrm{GL}_{n_4}(F) \times G'_0$.

Here, when $n_i = 0$, we understand that $\tau_i = \mathbf{1}_{\mathrm{GL}_0(F)}$.

2.3. Socle irreducible representations. Let G be again one of the split groups $\mathrm{GL}_n(F)$, $\mathrm{SO}_{2n+1}(F)$, or $\mathrm{Sp}_{2n}(F)$. For $\Pi \in \mathrm{Rep}(G)$, the *socle* of Π , denoted by $\mathrm{soc}(\Pi)$, is its maximal semisimple subrepresentation. The *cosocle* of Π is its maximal semisimple quotient and is denoted by $\mathrm{cosoc}(\Pi)$. We say that Π is *socle irreducible* (SI, for short) if $\mathrm{soc}(\Pi)$ is irreducible and occurs in $[\Pi]$ with multiplicity one.

Throughout this article, we will use the following properties of SI representations.

- (i) If $\Pi \in \mathrm{Rep}(G)$ is SI and $\mathrm{soc}(\Pi) \cong \mathrm{cosoc}(\Pi)$, then Π is irreducible.

- (ii) Suppose that $\rho \in \mathcal{C}^{\text{GL}}$ is not self-dual. For $\pi \in \text{Irr}(G)$, let $k \geq 0$ be an integer such that $D_\rho^{(k)}(\pi) \neq 0$ but $D_\rho^{(k+1)}(\pi) = 0$ (resp. $D_{\rho^\vee}^{(k)}(\pi) \neq 0$ but $D_{\rho^\vee}^{(k+1)}(\pi) = 0$). Then $\text{soc}(\rho \rtimes \pi)$ (resp. $\text{cosoc}(\rho \rtimes \pi)$) is an irreducible representation characterized by $D_\rho^{(k+1)}(\text{soc}(\rho \rtimes \pi)) = D_\rho^{(k)}(\pi)$ (resp. $D_{\rho^\vee}^{(k+1)}(\text{cosoc}(\rho \rtimes \pi)) = D_{\rho^\vee}^{(k)}(\pi)$).
- (iii) Assume that G is either split $\text{SO}_{2n+1}(F)$ or $\text{Sp}_{2n}(F)$. Let $S \in \text{Irr}^{\text{GL}}$ be a Speh representation whose cuspidal support consists of non-self-dual cuspidal representations. (For the notion of Speh representations, see Section 2.5 below.) Then for any $\pi \in \text{Irr}(G)$, the parabolic induction $S \rtimes \pi$ is SI. In particular, if $\rho \in \mathcal{C}^{\text{GL}}$ is not self-dual, then $\rho \rtimes \pi$ is SI for all $\pi \in \text{Irr}(G)$.

The property (i) is an easy exercise. On the other hand, (ii) and (iii) were proved in [10, Proposition 3.3] and [6, Proposition 3.4], respectively. They are analogues of the corresponding results for GL_n established in [30, Theorem 5.11, Proposition 6.6].

2.4. Representations of the Weil–Deligne–Arthur group. For $G = \text{GL}_n(F)$, $\text{SO}_{2n+1}(F)$, or $\text{Sp}_{2n}(F)$, the *Langlands dual group* $\widehat{G}$ is defined by

$$\widehat{G} = \begin{cases} \text{GL}_n(\mathbb{C}) & \text{if } G = \text{GL}_n(F), \\ \text{Sp}_{2n}(\mathbb{C}) & \text{if } G = \text{SO}_{2n+1}(F), \\ \text{SO}_{2n+1}(\mathbb{C}) & \text{if } G = \text{Sp}_{2n}(F). \end{cases}$$

We set N to be the dimension of the standard representation of $\widehat{G}$, i.e., $N = n$ (resp. $N = 2n$, $N = 2n + 1$) if $G = \text{GL}_n(F)$ (resp. $\text{SO}_{2n+1}(F)$, $G = \text{Sp}_{2n}(F)$).

Let W_F denote the Weil group of F . By abuse of language, we call a homomorphism

$$\psi: W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \rightarrow \text{GL}_N(\mathbb{C})$$

a *representation* of $W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})$ if $\psi(W_F)$ consists of semisimple elements, $\psi|_{W_F}$ is smooth and the restriction $\psi|_{\text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})}$ is algebraic. Two such representations with image in $\widehat{G}$ are said to be *equivalent* if they are conjugate under $\widehat{G}$, and we denote by $\Psi^+(G)$ the set of equivalence classes of such representations.

We say that:

- $\psi \in \Psi^+(G)$ is an *A-parameter* for G if $\psi(W_F)$ is bounded;
- $\psi \in \Psi^+(G)$ is an *L-parameter* for G if it is trivial on the second copy of $\text{SL}_2(\mathbb{C})$.

We write $\Psi(G)$ (resp. $\Phi(G)$) for the subset of $\Psi^+(G)$ consisting of *A-parameters* (resp. *L-parameters*). Through the embedding

$$W_F \times \text{SL}_2(\mathbb{C}) \hookrightarrow W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}), \quad (w, g) \mapsto (w, g, \mathbf{1}),$$

we identify each $\phi \in \Phi(G)$ with a homomorphism $\phi: W_F \times \text{SL}_2(\mathbb{C}) \rightarrow \widehat{G}$. We define the set of *tempered L-parameters* for G as

$$\Phi_{\text{temp}}(G) = \Phi(G) \cap \Psi(G).$$

In other words, $\phi \in \Phi(G)$ is tempered if and only if $\phi(W_F)$ is bounded.

By the local Langlands correspondence for $\text{GL}_n(F)$, established by Harris–Taylor [23] and Henniart [27], we can view each $\rho \in \mathcal{C}^{\text{GL}}$ as an irreducible representation of W_F . It has bounded image if and only if ρ is unitary, i.e., $e(\rho) = 0$.

Let S_a denote the unique irreducible algebraic representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension a . Then any representation ψ of $W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ can be decomposed as

$$\psi = \bigoplus_{i=1}^r \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}$$

for some $\rho_1, \dots, \rho_r \in \mathcal{C}^{\mathrm{GL}}$.

2.5. Representations of $\mathrm{GL}_n(F)$. A *segment* is a set of the form

$$[x, y]_\rho = \{\rho| \cdot |^x, \rho| \cdot |^{x-1}, \dots, \rho| \cdot |^y\},$$

where $\rho \in \mathcal{C}^{\mathrm{GL}}$ and $x, y \in \mathbb{R}$ with $x - y \in \mathbb{Z}_{\geq 0}$. One associates to it two irreducible representations $\Delta_\rho[x, y]$ and $Z_\rho[y, x]$ uniquely determined by the property that

$$\Delta_\rho[x, y] \hookrightarrow \rho| \cdot |^x \times \rho| \cdot |^{x-1} \times \dots \times \rho| \cdot |^y \twoheadrightarrow Z_\rho[y, x].$$

For $\rho \in \mathcal{C}^{\mathrm{GL}}$, $A, B \in \frac{1}{2}\mathbb{Z}$ with $A - B \in \mathbb{Z}$ and $-A \leq B \leq A$, the unique irreducible subrepresentation of

$$\Delta_\rho[B, -A] \times \Delta_\rho[B+1, -A+1] \times \dots \times \Delta_\rho[A, -B]$$

is denoted by

$$\mathrm{Sp}(\rho, a, b) = \begin{pmatrix} B & \dots & A \\ \vdots & \ddots & \vdots \\ -A & \dots & -B \end{pmatrix}_\rho,$$

where $a = A + B + 1$ and $b = A - B + 1$ denote respectively the number of rows and columns of the above matrix. We call it a *Speh representation*. It is unitary if and only if ρ is unitary. More generally, we write

$$\mathrm{Sp}(\rho, a, b)| \cdot |^s = \begin{pmatrix} B+s & \dots & A+s \\ \vdots & \ddots & \vdots \\ -A+s & \dots & -B+s \end{pmatrix}_\rho$$

for the corresponding (shifted) Speh representation.

By the local Langlands correspondence, there is a canonical bijection

$$\Phi(\mathrm{GL}_n(F)) \xrightarrow{1:1} \mathrm{Irr}(\mathrm{GL}_n(F)), \quad \phi \mapsto \tau_\phi.$$

If $\phi = \bigoplus_{i=1}^r \rho_i \boxtimes S_{a_i}$ with $e(\rho_1) \leq \dots \leq e(\rho_r)$, then τ_ϕ is the unique irreducible subrepresentation of the standard module

$$\lambda(\phi) = \Delta_{\rho_1}[x_1, -x_1] \times \dots \times \Delta_{\rho_r}[x_r, -x_r]$$

with $x_i = \frac{a_i-1}{2}$. In this case, we write

$$\tau_\phi = L(\Delta_{\rho_1}[x_1, -x_1], \dots, \Delta_{\rho_r}[x_r, -x_r]).$$

We will sometimes use the multisegment notation $\lambda(\mathbf{m}) = \lambda(\phi)$ and $L(\mathbf{m}) = \tau_\phi$, where

$$\mathbf{m} = [x_1, -x_1]_{\rho_1} + \dots + [x_r, -x_r]_{\rho_r}$$

is the associated multisegment. Here, a *multisegment* is a multiset of segments, viewed as a finite formal sum.

In particular, for $\psi = \bigoplus_{i=1}^r \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i} \in \Psi(\mathrm{GL}_n(F))$, the representation $\tau_\psi = \tau_{\phi_\psi}$ associated to the L -parameter

$$\phi_\psi: W_F \times \mathrm{SL}_2(\mathbb{C}) \ni (w, g) \mapsto \psi \left(w, g, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix} \right)$$

is the irreducible parabolic induction of unitary Speh representations:

$$\tau_\psi = \bigtimes_{i=1}^r \mathrm{Sp}(\rho_i, a_i, b_i).$$

For $\tau \in \mathrm{Irr}(\mathrm{GL}_n(F))$, its cuspidal support is denoted by $\mathrm{supp}(\tau)$. It is a multiset $\{\rho_1, \dots, \rho_t\}$ over $\mathcal{C}^{\mathrm{GL}}$ such that τ is a subquotient of $\rho_1 \times \dots \times \rho_t$.

2.6. Representations of classical groups. In this subsection, we let G be either split $\mathrm{SO}_{2n+1}(F)$ or $\mathrm{Sp}_{2n}(F)$. We regard $\psi \in \Psi^+(G)$ as a self-dual representation of $W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ of symplectic type or of orthogonal type. We decompose

$$\psi = \psi_{\mathrm{good}} \oplus \psi_{\mathrm{bad}}$$

such that for each irreducible summand $\rho \boxtimes S_a \boxtimes S_b$ of ψ , it is a summand of ψ_{good} if and only if $e(\rho) \in (1/2)\mathbb{Z}$ and $\rho| \cdot |^{-e(\rho)} \boxtimes S_{a+2|e(\rho)|} \boxtimes S_b$ is self-dual of the same type as ψ . We call ψ of *good parity* if $\psi = \psi_{\mathrm{good}}$. Notice that ψ_{bad} has a (non unique) decomposition of the form:

$$\psi_{\mathrm{bad}} = \psi_{\mathrm{bad}}^- \oplus (\psi_{\mathrm{bad}}^-)^\vee.$$

For an A -parameter $\psi \in \Psi(G)$, if $\psi_{\mathrm{good}} = \bigoplus_{i=1}^r \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}$, we set

$$A_\psi = \bigoplus_{i=1}^r \mathbb{Z}/2\mathbb{Z}e(\rho_i, a_i, b_i).$$

This is the free $\mathbb{Z}/2\mathbb{Z}$ -module with a canonical basis $\{e(\rho_i, a_i, b_i)\}_{i=1, \dots, r}$. Its quotient by the subgroup generated by

- $e(\rho_i, a_i, b_i) + e(\rho_j, a_j, b_j)$ such that $\rho_i \boxtimes S_{a_i} \boxtimes S_{b_i} \cong \rho_j \boxtimes S_{a_j} \boxtimes S_{b_j}$; and
- $e(\rho_1, a_1, b_1) + \dots + e(\rho_r, a_r, b_r)$

is denoted by $\mathcal{A}_\psi$. We often regard the Pontryagin dual $\widehat{\mathcal{A}_\psi}$ of $\mathcal{A}_\psi$ as a subgroup of the Pontryagin dual $\widehat{A_\psi}$ of A_ψ . For $\varepsilon \in \widehat{A_\psi}$, we set $\varepsilon(\rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}) = \varepsilon(e(\rho_i, a_i, b_i))$. If $\psi = \phi$ is a tempered L -parameter, we write $e(\rho_i, a_i) = e(\rho_i, a_i, 1)$ and $\varepsilon(\rho_i \boxtimes S_{a_i}) = \varepsilon(e(\rho_i, a_i))$.

By Arthur's endoscopic classification ([3, Theorem 2.2.1]), for $\psi \in \Psi(G)$, there is a multiset Π_ψ over $\mathrm{Irr}_{\mathrm{unit}}(G)$, called the A -packet associated to ψ , together with a map

$$\Pi_\psi \rightarrow \widehat{\mathcal{A}_\psi}, \pi \mapsto \langle \cdot, \pi \rangle_\psi$$

characterized by certain endoscopic character identities. We call an irreducible representation π of G of *Arthur type* if there is $\psi \in \Psi(G)$ such that $\pi \in \Pi_\psi$. In [35, 36, 37, 38, 39], Mœglin explicitly constructed the A -packet Π_ψ , and in particular, she showed in [39] that Π_ψ is multiplicity free, i.e., Π_ψ is a subset of $\mathrm{Irr}_{\mathrm{unit}}(G)$. For her construction, see also [52] or Section 5.1 below.

In the special case where $\psi = \phi$ is a tempered L -parameter, Arthur proves in [3, Theorem 2.2.1] that $\Pi_\phi \subset \mathrm{Irr}_{\mathrm{temp}}(G)$, that the map

$$\Pi_\phi \rightarrow \widehat{\mathcal{A}_\phi}$$

is bijective, and that

$$\mathrm{Irr}_{\mathrm{temp}}(G) = \bigsqcup_{\phi \in \Phi_{\mathrm{temp}}(G)} \Pi_{\phi}.$$

For $\varepsilon \in \widehat{\mathcal{A}}_{\phi}$, we denote by $\pi(\phi, \varepsilon)$ the corresponding element of Π_{ϕ} .

For tempered parameters, we also have the following useful property, which allows one to reduce certain problems to the good parity case. Let ϕ be a tempered L -parameter. Then ϕ decomposes as above as

$$\phi = \phi_{\mathrm{good}} \oplus \phi_{\mathrm{bad}}.$$

Choose a decomposition of $\phi_{\mathrm{bad}} = \phi_{\mathrm{bad}}^{-} \oplus (\phi_{\mathrm{bad}}^{-})^{\vee}$. Then by [3, Theorem 2.2.1], we have

$$\pi(\phi, \varepsilon) \cong \tau_{\phi_{\mathrm{bad}}^{-}} \rtimes \pi(\phi_{\mathrm{good}}, \varepsilon).$$

For general $\phi \in \Phi(G)$, write

$$\phi = \left(\bigoplus_{i=1}^r \rho_i \boxtimes S_{a_i} \right) \oplus \phi_0 \oplus \left(\bigoplus_{i=1}^r \rho_i^{\vee} \boxtimes S_{a_i} \right),$$

where $e(\rho_1) \leq \dots \leq e(\rho_r) < 0$ and $\phi_0 \in \Phi_{\mathrm{temp}}(G_0)$. Set $x_i = \frac{a_i-1}{2}$. Then for $\varepsilon \in \widehat{\mathcal{A}}_{\phi} = \widehat{\mathcal{A}}_{\phi_0}$, the Langlands subrepresentation theorem implies that the induced representation

$$\Delta_{\rho_1}[x_1, -x_1] \times \dots \times \Delta_{\rho_r}[x_r, -x_r] \rtimes \pi(\phi_0, \varepsilon)$$

is SI, and we denote its socle by $\pi(\phi, \varepsilon)$. Sometimes we also use multisegment notation and write

$$\pi(\phi, \varepsilon) = L(\mathbf{m}; \pi(\phi_0, \varepsilon)) = L(\Delta_{\rho_1}[x_1, -x_1], \dots, \Delta_{\rho_r}[x_r, -x_r]; \pi(\phi_0, \varepsilon)),$$

where

$$\mathbf{m} = [x_1, -x_1]_{\rho_1} + \dots + [x_r, -x_r]_{\rho_r}.$$

Define $\Pi_{\phi} = \{\pi(\phi, \varepsilon) \mid \varepsilon \in \widehat{\mathcal{A}}_{\phi}\}$. The Langlands classification allows one to extend Arthur's classification of tempered representations to all irreducible representations of G : Π_{ϕ} is a subset of $\mathrm{Irr}(G)$, the map

$$\Pi_{\phi} \rightarrow \widehat{\mathcal{A}}_{\phi}, \quad \pi(\phi, \varepsilon) \mapsto \varepsilon,$$

is bijective, and

$$\mathrm{Irr}(G) = \bigsqcup_{\phi \in \Phi(G)} \Pi_{\phi}.$$

We call Π_{ϕ} the L -packet associated to ϕ .

Let $\mathcal{C}_{\mathrm{unit}}^{\mathrm{GL}}$ (resp. $\mathcal{C}_{\mathrm{sd}}^{\mathrm{GL}}$) be the subset of $\mathcal{C}^{\mathrm{GL}}$ consisting of unitary (resp. self-dual) representations. For $\rho \in \mathcal{C}_{\mathrm{unit}}^{\mathrm{GL}}$, if we regard it as an irreducible bounded representation of W_F , then $\rho \boxtimes S_a \boxtimes S_b$ is self-dual of the same type as some (any) $\psi \in \Psi^+(G)$ if and only if one of the following holds:

- $\rho \in \mathcal{C}_{\mathrm{sd}}^{\mathrm{GL}}$, ρ is of the same type as some (any) $\psi \in \Psi^+(G)$, and $a + b$ is even;
- $\rho \in \mathcal{C}_{\mathrm{sd}}^{\mathrm{GL}}$, ρ is of the opposite type as some (any) $\psi \in \Psi^+(G)$, and $a + b$ is odd.

Moreover, by [51, Proposition 3.2], $\rho \in \mathcal{C}_{\mathrm{sd}}^{\mathrm{GL}}$ is of the same (resp. opposite) type as some (any) $\psi \in \Psi^+(G)$, if and only if for some (any) $\sigma \in \mathcal{C}^G$, there exists $m \in \mathbb{Z}$ (resp. $m \in (\frac{1}{2})\mathbb{Z} \setminus \mathbb{Z}$) such that $\rho \cdot |^m \rtimes \sigma$ is reducible.

Let $\mathcal{C}_{\text{good}}^{\text{GL}}$ be the subset of $\mathcal{C}^{\text{GL}}$ consisting of $\rho \in \mathcal{C}^{\text{GL}}$ such that $\rho| \cdot |^{m_\sigma} \rtimes \sigma$ is reducible for any $\sigma \in \mathcal{C}^G$ with an integer m_σ . Note that $\mathcal{C}_{\text{good}}^{\text{GL}}$ is preserved by the contragredient functor. By the cuspidal support theorem ([51, Proposition 8.1]), we obtain the following.

Proposition 2.3. *Let $\pi \in \text{Irr}(G)$. The following conditions are equivalent.*

- (1) *If $\pi \in \Pi_\phi$ for $\phi \in \Phi(G)$, then $\phi = \phi_{\text{good}}$.*
- (2) *If $\rho_1, \dots, \rho_r \in \mathcal{C}^{\text{GL}}$ and $\sigma \in \mathcal{C}^G$ satisfy*

$$\pi \hookrightarrow \rho_1 \times \cdots \times \rho_r \rtimes \sigma,$$

then $\rho_i \in \mathcal{C}_{\text{good}}^{\text{GL}}$ for each $1 \leq i \leq r$.

We say that $\pi \in \text{Irr}(G)$ is of *good parity* if any of the conditions of this proposition is satisfied. We denote by $\text{Irr}_{\text{good}}(G)$ (resp. $\text{Irr}_{\text{good}}^G$) the subset of $\text{Irr}(G)$ (resp. Irr^G) consisting of irreducible representations of good parity.

2.7. Methods for proving (non-)unitarity. Recall that an irreducible representation π of a group G is *Hermitian* if $\bar{\pi} \cong \pi^\vee$, where $\bar{\pi}$ denotes the complex conjugate of π . In other words, π is Hermitian if and only if it admits a non-degenerate G -invariant Hermitian form. Note that if π is unitary, then it is also Hermitian.

Now we list some basic methods for proving or disproving unitarity.

Proposition 2.4. *Fix a standard parabolic subgroup $P = MN$ of G with $M \cong \text{GL}_{m_1}(F) \times \cdots \times \text{GL}_{m_k}(F) \times G_0$. Let $\tau_i \in \text{Irr}(\text{GL}_{m_i}(F))$, $\pi_0 \in \text{Irr}(G_0)$ and set $\pi_M = \tau_1 \boxtimes \cdots \boxtimes \tau_k \boxtimes \pi_0 \in \text{Irr}(M)$.*

- (1) *(Unitary induction (UI)) If π_M is unitary, then $\text{Ind}_P^G(\pi_M)$ is a direct sum of irreducible unitary representations of G .*
- (2) *(Unitary reduction (UR)) If π_M is Hermitian and if $\text{Ind}_P^G(\pi_M)$ is irreducible and unitary, then π_M is also unitary.*
- (3) *(Complementary series (CS)) Let $x_1(t), \dots, x_k(t): [0, 1] \rightarrow \mathbb{R}$ be continuous functions, and set*

$$\Pi_t = \tau_1| \cdot |^{x_1(t)} \times \cdots \times \tau_k| \cdot |^{x_k(t)} \rtimes \pi_0.$$

If Π_t is irreducible and Hermitian for $0 \leq t < 1$, and if $\Pi_0 = \text{Ind}_P^G(\pi_M)$ is unitary, then all irreducible subquotients of Π_t are unitary for $0 \leq t \leq 1$.

- (4) *(Beyond the first reducibility point (RP1)) Suppose that*
 - $k = 1$;
 - $\tau_1 = \times_{i=1}^r \text{Sp}(\rho_i, c_i, d_i)$ is a product of unitary Speh representations with $\rho_i \in \mathcal{C}_{\text{sd}}^{\text{GL}}$ for $1 \leq i \leq r$;
 - π_0 is of Arthur type of good parity.

Let $\psi_M \in \Psi(M)$ be an A -parameter with $\pi_M \in \Pi_{\psi_M}$, and let $R_P(w, \pi_M, \psi_M)$ be the normalized intertwining operator defined by Arthur [3, Section 2.4] with $w \in W(M, G)$. Assume further that $R_P(w, \pi_M, \psi_M)$ is not a scalar (so in particular $\text{Ind}_P^G(\pi_M)$ is reducible). Then $\tau_1| \cdot |^s \rtimes \pi_0$ is not unitary for sufficiently small $s > 0$.

For (UI), (RP1) and (UR) (resp. (CS)), see [45] or [41, Section 2] (resp. [31, Lemma 3.3]). Finally, we record the following fundamental result.

Theorem 2.5 (J. Arthur [3]). *If $\pi \in \text{Irr}(G)$ is of Arthur type, then π is unitary.*

Arthur's proof relies on global methods. In the case of $\mathrm{GL}_n(F)$, representations of Arthur type are the products of representations of the form $\mathrm{Sp}(\rho, a, b)$ with $\rho \in \mathcal{C}_{\mathrm{unit}}^{\mathrm{GL}}$, and their unitarity can be established by purely local arguments (see [30, §6.2], and the references therein).

Question 2.6. *Can one prove the unitarity of an irreducible representation π of a classical group of Arthur type by purely local methods?*

While this question may be approachable for specific representations π on a case-by-case basis, we do not know of a general local proof.

3. THE SZ-DECOMPOSITION

From now on, we let G be either split $\mathrm{SO}_{2n+1}(F)$ or $\mathrm{Sp}_{2n}(F)$.

Definition 3.1. An irreducible representation π_0 of G of good parity is *almost cuspidal* if for any $\rho \in \mathcal{C}_{\mathrm{sd}}^{\mathrm{GL}}$ and $x \in \mathbb{R}$;

$$D_{\rho|\cdot|^x}(\pi_0) \neq 0 \implies x \in \{0, \frac{1}{2}\}.$$

The Langlands classification often allows one to reduce problems on irreducible representations to the case when the representation is tempered. However, for some problems, tempered representations are still difficult to handle. In this section, we introduce another decomposition of irreducible representations that will allow us to reduce our problem to almost cuspidal representations.

Proposition 3.2. *Let $\pi(\phi, \varepsilon) \in \mathrm{Irr}^G$. Then there is a quadruple*

$$(\tau_{\mathrm{bad}}^-, \{\tau_j^-\}_{1 \leq j \leq s}, \{\tau_j^+\}_{1 \leq j \leq r}, \{\pi_j\}_{0 \leq j \leq r+s}),$$

satisfying the following conditions.

- $\tau_{\mathrm{bad}}^- \in \mathrm{Irr}^{\mathrm{GL}}$, $\tau_j^\pm \in \mathrm{Irr}^{\mathrm{GL}}$ for all j , and $\pi_j \in \mathrm{Irr}_{\mathrm{good}}^G$ for all j ;
- there is a decomposition $\phi_{\mathrm{bad}} = \phi_{\mathrm{bad}}^- \oplus (\phi_{\mathrm{bad}}^-)^\vee$ such that τ_{bad}^- corresponds to ϕ_{bad}^- by the local Langlands correspondence;
- for each $1 \leq i \leq s$, there exist $\rho_i \in \mathcal{C}_{\mathrm{sd}}^{\mathrm{GL}}$, $y_i \leq x_i < 0$ and $k_i \in \mathbb{Z}_{>0}$ such that

$$\tau_{s-i+1}^- = \Delta_{\rho_i}[x_i, y_i]^{k_i}$$

and $\rho_i|\cdot|^{x_i} \in \mathcal{C}_{\mathrm{good}}^{\mathrm{GL}}$;

- for each $1 \leq i \leq r$, there exist $\rho'_i \in \mathcal{C}_{\mathrm{sd}}^{\mathrm{GL}}$, $y'_i \leq x'_i < 0$ and $k'_i \in \mathbb{Z}_{>0}$ such that

$$\tau_{r-i+1}^+ = Z_{\rho'_i}[-x'_i, -y'_i]^{k'_i}$$

and $\rho'_i|\cdot|^{x'_i} \in \mathcal{C}_{\mathrm{good}}^{\mathrm{GL}}$;

- $y_1 \leq \dots \leq y_s$, and if $i < j$ and $y_i = y_j$ then $x_i \leq x_j$;
- $y'_1 \leq \dots \leq y'_r$, and if $i < j$ and $y'_i = y'_j$ then $x'_i \leq x'_j$;
- π_0 is almost cuspidal;
- $\pi_i = \mathrm{soc}(\tau_i^+ \rtimes \pi_{i-1})$ for $1 \leq i \leq r$;
- $\pi_i = \mathrm{soc}(\tau_{i-r}^- \rtimes \pi_{i-1})$ for $r+1 \leq i \leq r+s$;
- $\pi(\phi_{\mathrm{good}}, \varepsilon) = \pi_{r+s}$ and $\pi(\phi, \varepsilon) = \mathrm{soc}(\tau_{\mathrm{bad}}^- \rtimes \pi_{r+s})$.

We call

$$(\tau_{\text{bad}}^-, \{\tau_j^-\}_{1 \leq j \leq s}, \{\tau_j^+\}_{1 \leq j \leq r}, \{\pi_j\}_{0 \leq j \leq r+s})$$

the *SZ-decomposition* of $\pi(\phi, \varepsilon)$.

Remark 3.3. See that if $(\tau_{\text{bad}}^-, \{\tau_j^-\}_{1 \leq j \leq s}, \{\tau_j^+\}_{1 \leq j \leq r}, \{\pi_j\}_{0 \leq j \leq r+s})$ is the SZ-decomposition of $\pi(\phi, \varepsilon)$, then we have embeddings

$$\begin{aligned} \pi(\phi, \varepsilon) &\hookrightarrow \tau_{\text{bad}}^- \rtimes \pi_{r+s} \\ &\hookrightarrow \tau_{\text{bad}}^- \times \tau_s^- \rtimes \pi_{r+s-1} \\ &\vdots \\ &\hookrightarrow \tau_{\text{bad}}^- \times \tau_s^- \times \cdots \times \tau_1^- \rtimes \pi_r \\ &\hookrightarrow \tau_{\text{bad}}^- \times \tau_s^- \times \cdots \times \tau_1^- \times \tau_r^+ \rtimes \pi_{r-1} \\ &\vdots \\ &\hookrightarrow \tau_{\text{bad}}^- \times \tau_s^- \times \cdots \times \tau_1^- \times \tau_r^+ \times \cdots \times \tau_1^+ \rtimes \pi_0. \end{aligned}$$

Moreover, $\tau_{\text{bad}}^- \times \tau_j^\pm$ is irreducible for all j by [53, Proposition 8.6].

Proof of Proposition 3.2. Let

$$\phi = \left(\bigoplus_{i=1}^a \rho_i | \cdot |^{\frac{x_i+y_i}{2}} \boxtimes S_{x_i-y_i+1} \right) \oplus \phi_0 \oplus \left(\bigoplus_{i=1}^a \rho_i^\vee | \cdot |^{-\frac{x_i+y_i}{2}} \boxtimes S_{x_i-y_i+1} \right)$$

be a decomposition of ϕ , where $\rho_i \in \mathcal{C}_{\text{unit}}^{\text{GL}}$, $x_i, y_i \in \mathbb{R}$ satisfy

$$x_1 + y_1 \leq \cdots \leq x_a + y_a < 0,$$

and $\phi_0 \in \Phi_{\text{temp}}(G_0)$. By the Langlands classification (see Paragraph 2.6),

$$\pi(\phi, \varepsilon) \cong \text{soc}(\Delta_{\rho_1}[x_1, y_1] \times \cdots \times \Delta_{\rho_a}[x_a, y_a] \rtimes \pi(\phi_0, \varepsilon)).$$

Choose a decomposition $\phi_{0,\text{bad}} = \phi_{0,\text{bad}}^- \oplus (\phi_{0,\text{bad}}^-)^\vee$, and write $\phi_{0,\text{bad}}^- = \bigoplus_{i=a+1}^b \rho_i \boxtimes S_{2x_i+1}$, where $\rho_i \in \mathcal{C}_{\text{unit}}^{\text{GL}}$ and $x_i \in (\frac{1}{2})\mathbb{Z}_{\geq 0}$ for $i = a+1, \dots, b$. Then

$$\pi(\phi_0, \varepsilon) \cong \Delta_{\rho_{a+1}}[x_{a+1}, -x_{a+1}] \times \cdots \times \Delta_{\rho_b}[x_b, -x_b] \rtimes \pi(\phi_{0,\text{good}}, \varepsilon).$$

Here, the irreducibility of the right-hand side follows from [3, Theorem 2.2.1]. Hence by setting $y_i = -x_i$ for $i = a+1, \dots, b$, we have

$$\pi(\phi, \varepsilon) \cong \text{soc}(\Delta_{\rho_1}[x_1, y_1] \times \cdots \times \Delta_{\rho_b}[x_b, y_b] \rtimes \pi(\phi_{0,\text{good}}, \varepsilon)).$$

Recall from [53, Theorem 9.7] that for $\rho, \rho' \in \mathcal{C}_{\text{unit}}^{\text{GL}}$, the induced representation

$$\Delta_\rho[x, y] \times \Delta_{\rho'}[x', y']$$

is reducible if and only if the two segments $[x, y]_\rho$ and $[x', y']_{\rho'}$ are *linked*, i.e., $\rho \cong \rho'$, $y \equiv y' \pmod{\mathbb{Z}}$, and $y < y' \leq x+1 < x'+1$ or $y' < y \leq x'+1 < x+1$. In particular, if

$$\rho_i | \cdot |^{\frac{x_i+y_i}{2}} \boxtimes S_{x_i-y_i+1} \subset \phi_{\text{good}} \quad \text{and} \quad \rho_j | \cdot |^{\frac{x_j+y_j}{2}} \boxtimes S_{x_j-y_j+1} \subset \phi_{\text{bad}},$$

then $\Delta_{\rho_i}[x_i, y_i] \times \Delta_{\rho_j}[x_j, y_j]$ is irreducible. Hence, after rearranging and relabelling the indices, we may rewrite $\pi(\phi, \varepsilon)$ as

$$\mathrm{soc} \left(\Delta_{\rho'_1}[x'_1, y'_1] \times \cdots \times \Delta_{\rho'_c}[x'_c, y'_c] \times \Delta_{\rho_1}[x_1, y_1] \times \cdots \times \Delta_{\rho_d}[x_d, y_d] \rtimes \pi(\phi_{0,\mathrm{good}}, \varepsilon) \right),$$

where $\rho'_1, \dots, \rho'_c, \rho_1, \dots, \rho_d \in \mathcal{C}_{\mathrm{unit}}^{\mathrm{GL}}$, $x_1 + y_1 \leq \cdots \leq x_d + y_d < 0$ and $x'_1 + y'_1 \leq \cdots \leq x'_c + y'_c \leq 0$, such that

$$\begin{aligned} \phi_{\mathrm{good}} &= \left(\bigoplus_{i=1}^d \rho_i \cdot \left| \cdot \right|^{\frac{x_i + y_i}{2}} \boxtimes S_{x_i - y_i + 1} \right) \oplus \phi_{0,\mathrm{good}} \oplus \left(\bigoplus_{i=1}^d \rho_i \cdot \left| \cdot \right|^{-\frac{x_i + y_i}{2}} \boxtimes S_{x_i - y_i + 1} \right), \\ \phi_{\mathrm{bad}}^- &= \bigoplus_{j=1}^c \rho'_j \cdot \left| \cdot \right|^{\frac{x'_j + y'_j}{2}} \boxtimes S_{x'_j - y'_j + 1}. \end{aligned}$$

If we denote by $\tau_{\mathrm{bad}}^- \in \mathrm{Irr}^{\mathrm{GL}}$ the irreducible representation corresponding to ϕ_{bad}^- , then

$$\tau_{\mathrm{bad}}^- = \mathrm{soc} \left(\Delta_{\rho'_1}[x'_1, y'_1] \times \cdots \times \Delta_{\rho'_c}[x'_c, y'_c] \right).$$

Hence $\pi(\phi, \varepsilon)$ is isomorphic to

$$\mathrm{soc} \left(\tau_{\mathrm{bad}}^- \times \Delta_{\rho_1}[x_1, y_1] \times \cdots \times \Delta_{\rho_d}[x_d, y_d] \rtimes \pi(\phi_{0,\mathrm{good}}, \varepsilon) \right).$$

Finally, by regrouping equal factors among the segment representations, we can rewrite

$$\pi(\phi, \varepsilon) = \mathrm{soc} \left(\tau_{\mathrm{bad}}^- \times \Delta_{\rho_1}[x_1, y_1]^{k_1} \times \cdots \times \Delta_{\rho_t}[x_t, y_t]^{k_t} \rtimes \pi(\phi_{0,\mathrm{good}}, \varepsilon) \right),$$

where

- $\rho_i \in \mathcal{C}_{\mathrm{unit}}^{\mathrm{GL}}$ and $x_i + y_i < 0$;
- $x_1 \leq \cdots \leq x_t$;
- $\pi(\phi_{\mathrm{good}}, \varepsilon) = \mathrm{soc} \left(\Delta_{\rho_1}[x_1, y_1]^{k_1} \times \cdots \times \Delta_{\rho_t}[x_t, y_t]^{k_t} \rtimes \pi(\phi_{0,\mathrm{good}}, \varepsilon) \right)$.

Here,

$$\Delta_{\rho_i}[x_i, y_i]^{k_i} = \underbrace{\Delta_{\rho_i}[x_i, y_i] \times \cdots \times \Delta_{\rho_i}[x_i, y_i]}_{k_i \text{ times}}.$$

Note that $\tau_{\mathrm{bad}}^- \times \Delta_{\rho_i}[x_i, y_i]$ is irreducible for each $i = 1, \dots, t$ by [53, Proposition 8.6].

Write

$$\{i \in \{1, \dots, t\} \mid x_i < 0\} = \{1, \dots, s\},$$

and set

$$\pi' = L \left(\Delta_{\rho_{s+1}}[x_{s+1}, y_{s+1}]^{k_{s+1}}, \dots, \Delta_{\rho_t}[x_t, y_t]^{k_t}; \pi(\phi_{0,\mathrm{good}}, \varepsilon) \right).$$

Note that for $\rho \in \mathcal{C}_{\mathrm{sd}}^{\mathrm{GL}}$ and $x \in \mathbb{R}$, by Corollary 2.2 and Casselman's criterion for tempered representations (see [28, Lemma 2.4]), we have

$$D_{\rho|\cdot|^x}(\pi') \neq 0 \implies x \geq 0.$$

After further rearranging, we can write

$$\pi(\phi, \varepsilon) = \mathrm{soc} \left(\tau_{\mathrm{bad}}^- \times \Delta_{\rho_1}[x_1, y_1]^{k_1} \times \cdots \times \Delta_{\rho_s}[x_s, y_s]^{k_s} \rtimes \pi' \right)$$

such that $y_1 \leq \cdots \leq y_s$, and such that if $i < j$ and $y_i = y_j$ then $x_i \leq x_j$. We set

$$\tau_{s-i+1}^- = \Delta_{\rho_i}[x_i, y_i]^{k_i}$$

for $i = 1, \dots, s$. As a conclusion, π is the socle of

$$\tau_{\text{bad}}^- \times \tau_s^- \times \cdots \times \tau_1^- \rtimes \pi'.$$

Next, let $\hat{\pi}'$ be the Aubert dual of π' (see [11]). Since π' is of good parity, so is $\hat{\pi}'$. As before, we may write $\hat{\pi}'$ in the form

$$\hat{\pi}' = L \left(\Delta_{\rho'_1}[x'_1, y'_1]^{k'_1}, \dots, \Delta_{\rho'_{t'}}[x'_{t'}, y'_{t'}]^{k'_{t'}}; \pi'_{\text{temp}} \right),$$

where $\rho'_i \in \mathcal{C}_{\text{unit}}^{\text{GL}}$, $x'_i + y'_i < 0$ and $x'_1 \leq \cdots \leq x'_{t'}$. Write $\{i \in \{1, \dots, t'\} \mid x'_i \leq -1\} = \{1, \dots, r\}$. We take $\pi_0 \in \text{Irr}^G$ such that

$$\hat{\pi}_0 = L \left(\Delta_{\rho'_{r+1}}[x'_{r+1}, y'_{r+1}]^{k'_{r+1}}, \dots, \Delta_{\rho'_{t'}}[x'_{t'}, y'_{t'}]^{k'_{t'}}; \pi'_{\text{temp}} \right).$$

Note that for $\rho \in \mathcal{C}_{\text{sd}}^{\text{GL}}$ and $x \in \mathbb{R}$, if $D_{\rho|\cdot|^x}(\hat{\pi}_0) \neq 0$, then by Corollary 2.2 and Casselman's criterion ([28, Lemma 2.4]), we have $x \geq -\frac{1}{2}$. Moreover, if such an x were positive, then since $\Delta_{\rho'_i}[x'_i, y'_i] \times \rho|\cdot|^x$ is irreducible for $1 \leq i \leq r$, we have $D_{\rho|\cdot|^x}(\hat{\pi}') \neq 0$. Hence $D_{\rho|\cdot|^{-x}}(\pi') \neq 0$, which is a contradiction. Therefore $x \in \{-\frac{1}{2}, 0\}$. By Aubert duality, we deduce that π_0 is almost cuspidal.

After further rearranging, we can write

$$\hat{\pi}' = \text{soc} \left(\Delta_{\rho'_1}[x'_1, y'_1]^{k'_1} \times \cdots \times \Delta_{\rho'_r}[x'_r, y'_r]^{k'_r} \rtimes \hat{\pi}_0 \right)$$

such that $y'_1 \leq \cdots \leq y'_r$, and such that if $i < j$ and $y'_i = y'_j$ then $x'_i \leq x'_j$. For $i = 1, \dots, r$, set

$$\tau_{r-i+1}^+ = Z_{\rho'_i}[-x'_i, -y'_i]^{k'_i}.$$

Applying again Aubert duality, we conclude that

$$\pi' = \text{soc} \left(\tau_r^+ \times \cdots \times \tau_1^+ \rtimes \pi_0 \right).$$

In particular, we have

$$\pi \hookrightarrow \tau_{\text{bad}}^- \times \tau_s^- \times \cdots \times \tau_1^- \times \tau_r^+ \times \cdots \times \tau_1^+ \rtimes \pi_0.$$

Finally, setting

$$\begin{aligned} \pi_1 &= \text{soc}(\tau_1^+ \rtimes \pi_0), \dots, \pi_r = \text{soc}(\tau_r^+ \rtimes \pi_{r-1}), \\ \pi_{r+1} &= \text{soc}(\tau_1^- \rtimes \pi_r), \dots, \pi_{r+s} = \text{soc}(\tau_s^- \rtimes \pi_{r+s-1}), \end{aligned}$$

we obtain the quadruple $(\tau_{\text{bad}}^-, \{\tau_j^-\}_{1 \leq j \leq s}, \{\tau_j^+\}_{1 \leq j \leq r}, \{\pi_j\}_{0 \leq j \leq r+s})$ as required. $\square$

4. THE MAIN THEOREMS

In this section, we state our main theorems and outline the strategy of their proofs. We continue to assume that G is either split $\text{SO}_{2n+1}(F)$ or $\text{Sp}_{2n}(F)$.

Definition 4.1. Let π be an irreducible representation of G . We say that π has *semisimple product with Speh representations* (SPS for short) if for every unitary Speh representations $S_1, \dots, S_k \in \text{Irr}^{\text{GL}}$, the induced representation

$$S_1 \times \cdots \times S_k \rtimes \pi$$

is semisimple.

Remark 4.2. If π is an irreducible unitary representation of G , then π has SPS. Indeed, for every unitary Speh representations $S_1, \dots, S_k \in \mathrm{Irr}^{\mathrm{GL}}$, the induced representation $S_1 \times \dots \times S_k \rtimes \pi$ is unitary (by Property (UI) in Proposition 2.4) and admissible, hence semisimple. This is the main property of unitary representations that we will use.

Our first main theorem is the following.

Theorem 4.3. *Let $\pi \in \mathrm{Irr}_{\mathrm{good}}(G)$. Then the following conditions are equivalent.*

- (1) π is unitary;
- (2) π is of Arthur type;
- (3) π has SPS.

Observe that (1) implies (3) by Remark 4.2, and (2) implies (1) by Arthur's result (Theorem 2.5). Therefore, it suffices to prove that (3) implies (2).

We will in fact prove a more general result, which we expect to be useful for the classification of the unitary dual of G .

Theorem 4.4. *Let $\pi(\phi, \varepsilon) \in \mathrm{Irr}^G$. If $\pi(\phi, \varepsilon)$ has SPS, then $\pi(\phi_{\mathrm{good}}, \varepsilon)$ is of Arthur type.*

The following corollary follows from Theorem 4.4 and Remark 4.2.

Corollary 4.5. *If $\pi(\phi, \varepsilon)$ is an irreducible unitary representation of G , then $\pi(\phi_{\mathrm{good}}, \varepsilon)$ is of Arthur type.*

Theorem 4.4 is an immediate consequence of the next two theorems.

Theorem 4.6. *Let $\pi \in \mathrm{Irr}^G$ be an almost cuspidal representation. Then π is of Arthur type.*

Theorem 4.7. *Let $\pi(\phi, \varepsilon) \in \mathrm{Irr}^G$, and let*

$$(\tau_{\mathrm{bad}}^-, \{\tau_j^-\}_{1 \leq j \leq s}, \{\tau_j^+\}_{1 \leq j \leq r}, \{\pi_j\}_{0 \leq j \leq r+s})$$

be the SZ-decomposition of $\pi(\phi, \varepsilon)$. If $\pi(\phi, \varepsilon)$ has SPS and π_0 is of Arthur type, then π_i is also of Arthur type for any $i \in \{1, \dots, r+s\}$. In particular, in this case, $\pi(\phi_{\mathrm{good}}, \varepsilon) = \pi_{r+s}$ is of Arthur type.

In the proofs of Theorems 4.6 and 4.7, we need to show that certain representations are of Arthur type. For this purpose, we use Mœglin's local construction of representations of Arthur type, as refined by the first-named author. This will be recalled in the next section. We will then prove Theorems 4.6 and 4.7 in Sections 6 and 7, respectively.

5. MÖGLIN'S CONSTRUCTION

In this section, we review Mœglin's explicit construction of local A -packets, as refined by the first author in [5]. To prove Theorem 4.7, we will use the criterion obtained in [7] for deciding whether a given $\pi \in \mathrm{Irr}_{\mathrm{good}}^G$ is of Arthur type. We will explain this criterion in Section 5.2.

5.1. Extended multisegments. The following notion was introduced in [5].

Definition 5.1. (1) An *extended segment* is a triple $([A, B]_{\rho}, l, \eta)$, where

- $[A, B]_{\rho}$ is a segment with $A, B \in \frac{1}{2}\mathbb{Z}$ (so that $A \geq B$ and $A - B \in \mathbb{Z}$);
- $l \in \mathbb{Z}$ satisfies $0 \leq l \leq \frac{b}{2}$, where $b = A - B + 1$;
- $\eta \in \{\pm 1\}$.

- (2) When b is even and $l = \frac{b}{2}$, the sign η is disregarded. Namely, we consider the equivalence classes with respect to the following equivalence relation. Two extended segments $([A, B]_\rho, l, \eta)$ and $([A', B']_{\rho'}, l', \eta')$ are said to be *equivalent* if

- $[A, B]_\rho = [A', B']_{\rho'}$;
- $l = l'$; and
- $\eta = \eta'$ if $l = l' < \frac{b}{2}$.

Two multisets of extended segments $\{([A_i, B_i]_{\rho_i}, l_i, \eta_i)\}_{i \in I}$ and $\{([A'_i, B'_i]_{\rho'_i}, l'_i, \eta'_i)\}_{i \in I}$ with the same index set I are said to be *equivalent* if $([A_i, B_i]_{\rho_i}, l_i, \eta_i)$ and $([A'_i, B'_i]_{\rho'_i}, l'_i, \eta'_i)$ are equivalent for all $i \in I$.

- (3) An *extended multisegment* for G is an equivalence class of multisets of extended segments

$$\mathcal{E} = \{([A_i, B_i]_{\rho_i}, l_i, \eta_i)\}_{i \in (I, >)}$$

such that

- $\rho_i \in \mathcal{C}_{\text{sd}}^{\text{GL}}$;
- $A_i + B_i \geq 0$;
- $\psi_{\mathcal{E}} = \bigoplus_{i \in I} \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}$ is an A -parameter for G of good parity, where $a_i = A_i + B_i + 1$ and $b_i = A_i - B_i + 1$;
- the sign condition

$$\prod_{i \in I} (-1)^{\lfloor \frac{b_i}{2} \rfloor + l_i} \eta_i^{b_i} = 1$$

holds;

- $>$ is a total order on I , which is *admissible* in the sense that

$$\rho_i \cong \rho_j, A_i < A_j, B_i < B_j \implies i < j$$

and

$$\rho_i \cong \rho_j, B_i < B_j < 0 \implies i < j.$$

Let $\mathcal{E} = \{([A_i, B_i]_{\rho_i}, l_i, \eta_i)\}_{i \in (I, >)}$ be an extended multisegment for G . When $i < j$, we write

$$([A_i, B_i]_{\rho_i}, l_i, \eta_i) < ([A_j, B_j]_{\rho_j}, l_j, \eta_j).$$

For simplicity, we write $I = \{1, \dots, m\}$ with the usual order. For $\rho \in \mathcal{C}_{\text{sd}}^{\text{GL}}$, set

$$\mathcal{E}_\rho = \{([A_i, B_i]_{\rho_i}, l_i, \eta_i) \mid i \in (I, >), \rho_i \cong \rho\}.$$

We identify $\mathcal{E}_\rho$ with the following symbol. When $\mathcal{E}_\rho = \{([A, B]_\rho, l, \eta)\}$ is a singleton, we write

$$\mathcal{E}_\rho = \left(\underbrace{\triangleleft \triangleleft \dots \triangleleft}_{l} \quad \overset{B+l}{\odot} \odot \dots \odot \overset{A-l}{\odot} \quad \underbrace{\triangleright \dots \triangleright}_{l} \overset{A}{\triangleright} \right)_\rho,$$

where $\odot$ is replaced alternately by $\oplus$ and $\ominus$, and the leftmost $\odot$ is replaced by $\oplus$ if $\eta = +1$ (resp. by $\ominus$ if $\eta = -1$). In general, we place the symbols vertically.

Example 5.2. When $\mathcal{E}_\rho = \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in \{1 < 2 < 3 < 4\}}$ with

- $[A_1, B_1]_\rho = [3, 1]_\rho$, $[A_2, B_2]_\rho = [5, 2]_\rho$, $[A_3, B_3]_\rho = [6, 3]_\rho$, and $[A_4, B_4]_\rho = [5, 4]_\rho$;
- $(l_1, l_2, l_3, l_4) = (0, 1, 2, 0)$ and $(\eta_1, \eta_2, \eta_4) = (-1, -1, -1)$,

the symbol is

$$\mathcal{E}_\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \ominus & \oplus & \ominus & & & \\ & \triangleleft & \ominus & \oplus & \triangleright & \\ & & \triangleleft & \triangleleft & \triangleright & \triangleright \\ & & & \ominus & \oplus & \end{pmatrix}_\rho.$$

This symbol does not depend on η_3 .

We say that $\mathcal{E} = \{([A_i, B_i]_{\rho_i}, l_i, \eta_i)\}_{i \in (I, >)}$ has a *non-negative discrete diagonal restriction (DDR)* if

$$0 \leq B_1 \leq A_1 < B_2 \leq A_2 < \cdots < B_m \leq A_m.$$

In this case, we define $\pi(\mathcal{E}) \in \mathrm{Irr}(G)$ by

$$\pi(\mathcal{E}) = \mathrm{soc} \left(\bigotimes_{i=1}^m \begin{pmatrix} B_i & \cdots & B_i + l_i - 1 \\ \vdots & \ddots & \vdots \\ -A_i & \cdots & -(A_i - l_i + 1) \end{pmatrix}_{\rho_i} \rtimes \pi(\phi, \varepsilon) \right)$$

with

$$\phi = \bigoplus_{i=1}^m \bigoplus_{j=0}^{A_i - B_i - 2l_i} \rho_i \boxtimes S_{2(B_i + l_i + j) + 1}$$

and $\varepsilon(\rho_i \boxtimes S_{2(B_i + l_i + j) + 1}) = (-1)^j \eta_i$ for $1 \leq i \leq m$ and $0 \leq j \leq A_i - B_i - 2l_i$. In particular, if $\mathcal{E} = \mathcal{E}_\rho = \{([A, B]_\rho, l, +1)\}$ is a singleton, then

$$\begin{aligned} \pi \left(\underbrace{\begin{pmatrix} B & \cdots & B+l-1 \\ \triangleleft & \triangleleft & \triangleleft \end{pmatrix}}_l \oplus \begin{pmatrix} B+l & \cdots & A-l \\ \oplus & \ominus & \odot \end{pmatrix} \oplus \underbrace{\begin{pmatrix} A-l+1 & \cdots & A \\ \triangleright & \cdots & \triangleright \end{pmatrix}}_l \right)_\rho \\ = L(\Delta_\rho[B, -A], \Delta_\rho[B+1, -A+1], \dots, \Delta_\rho[B+l-1, -A+l-1]; \pi(\phi, \varepsilon)) \end{aligned}$$

with $\phi = \rho \boxtimes (S_{2(B+l)+1} \oplus \cdots \oplus S_{2(A-l)+1})$ and

$$\varepsilon(\rho \boxtimes (S_{2(B+l)+1})) = 1, \varepsilon(\rho \boxtimes (S_{2(B+l+1)+1})) = -1, \dots, \varepsilon(\rho \boxtimes (S_{2(A-l)+1})) = (-1)^{A-B}.$$

In the general case, given an extended multisegment

$$\mathcal{E} = \{([A_i, B_i]_{\rho_i}, l_i, \eta_i)\}_{i \in (I, >)},$$

we define the representation $\pi(\mathcal{E})$ as follows. Choose a sequence of non-negative integers $\{t_i\}_{i \in (I, >)}$ such that

$$\mathcal{E}_{\gg} = \{([A_i + t_i, B_i + t_i]_{\rho_i}, l_i, \eta_i)\}_{i \in (I, >)}$$

has a non-negative DDR, and define

$$\pi(\mathcal{E}) = \bigcirc_{1 \leq i \leq m} \left(D_{\rho_i | \cdot |^{B_i+1}, \dots, \rho_i | \cdot |^{A_i+1}} \circ \cdots \circ D_{\rho_i | \cdot |^{B_i+t_i}, \dots, \rho_i | \cdot |^{A_i+t_i}} \right) (\pi(\mathcal{E}_{\gg})).$$

Here, $\bigcirc_{1 \leq i \leq m} f_i$ means that $f_m \circ \cdots \circ f_1$. This definition does not depend on the choice of $\{t_i\}_{i \in (I, >)}$. Then $\pi(\mathcal{E})$ is either irreducible or zero. Moreover, there is a criterion for determining whether $\pi(\mathcal{E}) \neq 0$ (see [5, Theorems 1.3, 1.4]). The next theorem describes the A -packet Π_ψ in terms of extended multisegments.

Theorem 5.3 ([5, Theorem 1.2]). *For $\psi \in \Psi(G)$ of good parity, we have*

$$\Pi_\psi = \{\pi(\mathcal{E}) \mid \psi_\mathcal{E} \cong \psi\} \setminus \{0\}.$$

5.2. Algorithm for Arthur type representations. In [7, Algorithm 1.3], the first author gave an algorithm for determining whether an irreducible representation of G (of good parity) is of Arthur type. It relies on the following result (see [7, Theorem 4.1]).

Theorem 5.4. *Let $\pi \in \text{Irr}^G$, and let*

$$(\tau_{\text{bad}}^-, \{\tau_j^-\}_{1 \leq j \leq s}, \{\tau_j^+\}_{1 \leq j \leq r}, \{\pi_j\}_{0 \leq j \leq r+s})$$

be its SZ-decomposition. Fix $1 \leq i \leq r$, and write

$$\tau_i^+ = Z_\rho[B+1, A+1]^k.$$

*Then π_i is of Arthur type if and only if π_{i-1} is of Arthur type and is of the form $\pi(\mathcal{E}^+)$, where the largest k elements of $\mathcal{E}^+$, with respect to the total order $>$ on I , are of the form $([A, B]_\rho, *, *)$. Similarly, fix $1 \leq j \leq s$, and write*

$$\tau_j^- = \Delta_\rho[B-1, -A-1]^k.$$

*Then π_{r+j} is of Arthur type if and only if π_{r+j-1} is of Arthur type and is of the form $\pi(\mathcal{E}^-)$, where the lowest k elements of $\mathcal{E}^-$, with respect to the total order $>$ on I , are of the form $([A, B]_\rho, *, *)$.*

We will prove in the next section that π_0 is always of Arthur type. Moreover, by [7, Theorem 3.5], if π_{i-1} (resp. π_{r+j-1}) is of Arthur type, then one can list all extended multisegments $\mathcal{E}^+$ (resp. $\mathcal{E}^-$) such that $\pi_{i-1} \cong \pi(\mathcal{E}^+)$ (resp. $\pi_{r+j-1} \cong \pi(\mathcal{E}^-)$). Therefore, it is in principle decidable whether π_{r+s} is of Arthur type.

6. THE FIRST INDUCTION STEP

In this section, we will prove Theorem 4.6. Indeed, we will prove a slightly more general statement to facilitate the inductive argument in its proof. To state it, we need to introduce some notation.

Let $\pi \in \text{Irr}_{\text{good}}^G$. Write

$$\pi = L(\Delta_{\rho_1}[x_1, -y_1], \dots, \Delta_{\rho_r}[x_r, -y_r]; \pi(\phi, \varepsilon))$$

as in the Langlands classification, where

$$\phi = \bigoplus_{j=1}^s \rho'_j \boxtimes S_{2z_j+1}$$

is a tempered L -parameter such that $\rho_1, \dots, \rho_r, \rho'_1, \dots, \rho'_s \in \mathcal{C}_{\text{sd}}^{\text{GL}}$. Fix $\rho \in \mathcal{C}_{\text{sd}}^{\text{GL}}$. We set

$$\begin{aligned} L_{\rho|\cdot|x}(\pi) &= L_x(\pi) = \{i \in \{1, \dots, r\} \mid \rho_i \cong \rho, x_i = x\}, \\ R_{\rho|\cdot|x}(\pi) &= R_x(\pi) = \{i \in \{1, \dots, r\} \mid \rho_i \cong \rho, y_i = x\}, \\ T_{\rho|\cdot|x}(\pi) &= T_x(\pi) = \{j \in \{1, \dots, s\} \mid \rho'_j \cong \rho, z_j = x\} \end{aligned}$$

and

$$A_{\rho|\cdot|x}(\pi) = A_x(\pi) = L_x \sqcup R_x \sqcup T_x.$$

Note that if $D_{\rho|\cdot|x}(\pi) \neq 0$, then $A_{\rho|\cdot|x}(\pi) \neq \emptyset$ by Corollary 2.2 and [4, Lemma 4.1], but the converse is not true in general. Indeed, the precise formula for the higher derivatives $D_{\rho|\cdot|x}^{(k)}(\pi)$ involves the best matching between certain subsets of $A_{\rho|\cdot|x}(\pi)$ and $A_{\rho|\cdot|x-1}(\pi)$. For details, see [10, Theorem 7.1].

Motivated by Mœglin's construction of local A -packets for non-negative DDR parameters, we will prove the following result.

Theorem 6.1. *Keep the notations as above. For each $\rho \in \mathcal{C}_{\mathrm{sd}}^{\mathrm{GL}}$, let $b_\rho \in \{0, \frac{1}{2}\}$ such that $\rho \boxtimes S_{2b_\rho+1}$ is self-dual of the same type as ϕ . Assume that there exist half-integers $a_\rho, a_1, \dots, a_k, b_1, \dots, b_k$ (depending on ρ) such that*

- they are all congruent to b_ρ modulo $\mathbb{Z}$;
- we have

$$b_\rho \leq a_\rho < b_k \leq a_k < b_{k-1} \leq a_{k-1} < \dots < b_1 \leq a_1;$$

- if $D_{\rho|\cdot|}^x(\pi) \neq 0$, then $x \in \{b_\rho, b_k, \dots, b_1\}$;
- if $A_x(\pi) \neq \emptyset$, then $b_\rho \leq x \leq a_\rho$ or $b_i \leq x \leq a_i$ for some $1 \leq i \leq k$;
- if $b_i \leq x \leq a_i$ for some $1 \leq i \leq k$, then $|A_x(\pi)| = 1$, so that $A_x(\pi)$ coincides with exactly one of $L_x(\pi)$, $T_x(\pi)$ or $R_x(\pi)$;
- for each $1 \leq i \leq k$,

$$|\{x \mid b_i \leq x \leq a_i, A_x(\pi) = L_x(\pi)\}| = |\{x \mid b_i \leq x \leq a_i, A_x(\pi) = R_x(\pi)\}|.$$

Then π is of Arthur type.

If $\pi \in \mathrm{Irr}_{\mathrm{good}}^G$ is almost cuspidal, then π satisfies the conditions of Theorem 6.1 with $k = 0$, and hence π is of Arthur type. Thus Theorem 6.1 implies Theorem 4.6.

We prove Theorem 6.1 by induction on

$$X(\pi) = \sum_{\rho \in \mathcal{C}_{\mathrm{sd}}^{\mathrm{GL}}} \sum_{b_\rho < x \leq a_\rho} |A_{\rho|\cdot|}^x(\pi)|.$$

Suppose first that this number is zero. Then there are $\eta_\rho, \eta_i \in \{\pm 1\}$ (which depend on ρ) such that if we set

$$\mathcal{E} = \bigcup_{\rho \in \mathcal{C}_{\mathrm{sd}}^{\mathrm{GL}}} \left(\underbrace{\{([b_\rho, b_\rho]_\rho, 0, \eta_\rho), \dots, ([b_\rho, b_\rho]_\rho, 0, \eta_\rho)\}}_{|A_{b_\rho}(\pi)|} \cup \bigcup_{i=1}^k \{([a_i, b_i]_\rho, l_i, \eta_i)\} \right)$$

with

$$l_i = |\{x \mid b_i \leq x \leq a_i, A_x(\pi) = L_x(\pi)\}| = |\{x \mid b_i \leq x \leq a_i, A_x(\pi) = R_x(\pi)\}|,$$

then we have $\pi(\mathcal{E}) = \pi$. See Section 5.1. Therefore, π is of Arthur type.

Now suppose that $A_{a_\rho}(\pi) \neq \emptyset$ for some $\rho \in \mathcal{C}_{\mathrm{sd}}^{\mathrm{GL}}$ and $a_\rho > b_\rho$. Replacing π with

$$\mathrm{soc}(Z_\rho[b_k, a_k] \rtimes (\dots \rtimes \mathrm{soc}(Z_\rho[b_2 + 1, a_2 + 1] \rtimes \mathrm{soc}(Z_\rho[b_1 + 1, a_1 + 1] \rtimes \pi)) \dots))$$

has the effect of replacing (a_i, b_i) with $(a_i + 1, b_i + 1)$ for all $1 \leq i \leq k$. Repeating this procedure, we may thus assume that b_k is sufficiently large so that there exists an integer $t > 0$ such that

$$a_\rho + 1 < b_\rho + t \leq a_\rho + t < b_k - 1.$$

Here, when $k = 0$, the last inequality is omitted. Consider

$$\pi' = \mathrm{soc} \left(\begin{pmatrix} b_\rho + t & \dots & a_\rho + t \\ \vdots & \ddots & \vdots \\ b_\rho + 1 & \dots & a_\rho + 1 \end{pmatrix}_\rho \rtimes \pi \right)$$

which is irreducible by [6, Proposition 3.4]. By Tadić's formula (Corollary 2.2), we see that

$$D_{\rho|\cdot| x}(\pi') \neq 0 \implies x \in \{b_\rho, b_\rho + t, b_k, \dots, b_1, -(a_\rho + 1)\}.$$

Lemma 6.2. *In the setting above, we have*

$$D_{\rho|\cdot| -(a_\rho+1)}(\pi') = 0.$$

Proof. Suppose that $D_{\rho|\cdot| -(a_\rho+1)}(\pi') \neq 0$. Then the composition map

$$\begin{aligned} \pi' &\hookrightarrow \begin{pmatrix} b_\rho + t & \dots & a_\rho + t \\ \vdots & \ddots & \vdots \\ b_\rho + 2 & \dots & a_\rho + 2 \end{pmatrix}_\rho \times Z_\rho[b_\rho + 1, a_\rho] \times \rho|\cdot|^{a_\rho+1} \rtimes \pi \\ &\twoheadrightarrow \begin{pmatrix} b_\rho + t & \dots & a_\rho + t \\ \vdots & \ddots & \vdots \\ b_\rho + 2 & \dots & a_\rho + 2 \end{pmatrix}_\rho \times Z_\rho[b_\rho + 1, a_\rho] \rtimes \text{soc}(\rho|\cdot|^{-(a_\rho+1)} \rtimes \pi) \end{aligned}$$

is nonzero. Hence

$$\pi = D_{\rho|\cdot| -(a_\rho+1)} \circ D_{\rho|\cdot|^{b_\rho+1}, \dots, \rho|\cdot|^{a_\rho}} \circ \left(D_{\rho|\cdot|^{b_\rho+2}, \dots, \rho|\cdot|^{a_\rho+2}} \circ \dots \circ D_{\rho|\cdot|^{b_\rho+t}, \dots, \rho|\cdot|^{a_\rho+t}} \right) (\pi').$$

On the other hand, by definition of π' , we have

$$\pi = D_{\rho|\cdot|^{a_\rho+1}} \circ D_{\rho|\cdot|^{b_\rho+1}, \dots, \rho|\cdot|^{a_\rho}} \circ \left(D_{\rho|\cdot|^{b_\rho+2}, \dots, \rho|\cdot|^{a_\rho+2}} \circ \dots \circ D_{\rho|\cdot|^{b_\rho+t}, \dots, \rho|\cdot|^{a_\rho+t}} \right) (\pi').$$

By the property (ii) in Section 2.3, these identities imply that

$$\text{soc}(\rho|\cdot|^{a_\rho+1} \rtimes \pi) \cong \text{cosoc}(\rho|\cdot|^{a_\rho+1} \rtimes \pi),$$

and hence that $\rho|\cdot|^{a_\rho+1} \rtimes \pi$ is irreducible by (i) in the same subsection. But this contradicts [10, Corollary 7.2], since $A_{a_\rho}(\pi) \neq \emptyset$ whereas $A_{a_\rho+1}(\pi) = \emptyset$. $\square$

Now we consider $A_x(\pi')$.

Lemma 6.3. *We have*

$$|A_x(\pi')| = \begin{cases} |A_x(\pi)| - 1 & \text{if } b_\rho \leq x \leq a_\rho, \\ 1 & \text{if } b_\rho + t \leq x \leq a_\rho + t, \\ 1 & \text{if } b_i \leq x \leq a_i, \ 1 \leq i \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for $b_\rho + t \leq x \leq a_\rho + t$, exactly one of

$$A_x(\pi') = L_x(\pi'), \quad A_x(\pi') = T_x(\pi'), \quad \text{or} \quad A_x(\pi') = R_x(\pi')$$

holds.

Proof. Write $\pi' = L(\mathbf{m}'; \pi(\phi', \varepsilon'))$. Since

$$D_{\rho|\cdot| x}(\pi') \neq 0 \implies x \in \{b_\rho, b_\rho + t, b_k, \dots, b_1\},$$

by looking at the cuspidal support, we see that there is $0 \leq l \leq (a_\rho - b_\rho + 1)/2$ such that

- $\sum_{j=0}^{l-1} [b_\rho + t + j, -a_\rho - t + j]_\rho \leq \mathbf{m}'$;
- $\bigoplus_{j=0}^{a_\rho - b_\rho - 2l} \rho \boxtimes S_{2(b_\rho + t + l + j) + 1} \subset \phi'$;

- the signs $(\varepsilon'(\rho \boxtimes S_{2(b_\rho+t+l+j)+1}))_{0 \leq j \leq a_\rho - b_\rho - 2l}$ are alternating.

Then by [10, Theorem 7.1], we have

$$|A_x(D_{\rho|\cdot|b_\rho+t, \dots, \rho|\cdot|b_\rho+1}(\pi'))| - |A_x(\pi')| = \begin{cases} 1 & \text{if } x = b_\rho, \\ -1 & \text{if } x = b_\rho + t, \\ 0 & \text{otherwise.} \end{cases}$$

Repeating this observation, we have

$$\begin{aligned} & |A_x(D_{\rho|\cdot|a_\rho+t, \dots, \rho|\cdot|a_\rho+1} \circ \dots \circ D_{\rho|\cdot|b_\rho+t, \dots, \rho|\cdot|b_\rho+1}(\pi'))| - |A_x(\pi')| \\ &= \begin{cases} 1 & \text{if } b_\rho \leq x \leq a_\rho \\ -1 & \text{if } b_\rho + t \leq x \leq a_\rho + t, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Since

$$D_{\rho|\cdot|a_\rho+t, \dots, \rho|\cdot|a_\rho+1} \circ \dots \circ D_{\rho|\cdot|b_\rho+t, \dots, \rho|\cdot|b_\rho+1}(\pi') \cong \pi,$$

we obtain the assertion. $\square$

Since $D_{\rho|\cdot|x}(\pi') = 0$ for $b_\rho + t < x \leq a_\rho + t$, by [10, Theorem 7.1], we see that

- if $x \geq b_\rho + t + 1$ and $A_x(\pi') = L_x(\pi')$, then $A_{x-1}(\pi') = L_{x-1}(\pi')$;
- if $x \leq a_\rho + t - 1$ and $A_x(\pi') = R_x(\pi')$, then $A_{x+1}(\pi') = R_{x+1}(\pi')$.

Write

$$\begin{aligned} l_+ &= |\{b_\rho + t \leq x \leq a_\rho + t \mid A_x(\pi') = L_x(\pi')\}|, \\ l_- &= |\{b_\rho + t \leq x \leq a_\rho + t \mid A_x(\pi') = R_x(\pi')\}|. \end{aligned}$$

A key lemma is the following.

Lemma 6.4. *We have $l_+ = l_-$.*

Proof. We first show that $l_+ \leq l_-$. Indeed, let $x \geq b_\rho + t$. If $A_x(\pi') = L_x(\pi')$, then $\Delta_\rho[x, -y]$ appears in the Langlands data of π' for some $y > x$. Hence $A_y(\pi') = R_y(\pi')$. The map $x \mapsto y$ induces an injection

$$\{x \geq b_\rho + t \mid A_x(\pi') = L_x(\pi')\} \hookrightarrow \{y \geq b_\rho + t \mid A_y(\pi') = R_y(\pi')\}.$$

Since

$$|\{x \mid b_i \leq x \leq a_i, A_x(\pi) = L_x(\pi)\}| = |\{y \mid b_i \leq y \leq a_i, A_y(\pi) = R_y(\pi)\}|$$

for $1 \leq i \leq k$ by assumption, we have $l_+ \leq l_-$. In particular, if $l_- = 0$, then $l_+ = 0$ as well and we get $l_+ = l_-$.

In the rest of the proof of this lemma, we assume that $l_- > 0$. In particular, $A_{a_\rho+t}(\pi') = R_{a_\rho+t}(\pi')$. It implies by [10, Corollary 7.2] that $L_{a_\rho}(\pi) = T_{a_\rho}(\pi) = \emptyset$. Since $D_{\rho|\cdot|x}(\pi) = 0$ for any $b_\rho < x \leq a_\rho$, we see that $\Delta_\rho[b_\rho, -a_\rho]$ appears in the Langlands data of π with multiplicity exactly $k = |R_{a_\rho}(\pi)| = |A_{a_\rho}(\pi)|$. By [10, Theorem 7.1], if $\Delta_\rho[b'_\rho + t, -a_\rho - t]$ appears in the Langlands data of π' for some b'_ρ , then $\Delta_\rho[b'_\rho, -a_\rho]$ appears in those of π . Hence $b'_\rho = b_\rho$ so that the singleton $A_{b_\rho+t}(\pi')$ is equal to $L_{b_\rho+t}(\pi')$.

If $A_y(\pi') = R_y(\pi')$ for some $b_\rho + t < y < a_\rho + t$, then the Langlands data of π' contains $\Delta_\rho[x, -y]$ for some $b_\rho \leq x < y$. We claim that $x > b_\rho + t$. For the sake of a contradiction, we assume that there is $\Delta_\rho[x, -y]$ in the Langlands data of π' such that $x \leq b_\rho + t$ and

$b_\rho + t < y < a_\rho + t$. We may assume that y is maximal subject to this condition. Then $\Delta_\rho[x', -(y+1)]$ is in the Langlands data of π' for some $x' \geq b_\rho + t$. Since the segment $[x, -y]_\rho$ is contained in $[x', -(y+1)]_\rho$, by [10, Theorem 7.1], we have $D_{\rho|\cdot|y+1}(\pi') \neq 0$, which is a contradiction. Therefore, for $b_\rho + t \leq y \leq a_\rho + t$ with $A_y(\pi') = R_y(\pi')$, if $\Delta_\rho[x, -y]$ is in the Langlands data of π' , then $b_\rho + t \leq x \leq a_\rho + t$. This gives the bijection

$$\{b_\rho + t \leq y \leq a_\rho + t \mid A_y(\pi') = R_y(\pi')\} \rightarrow \{b_\rho + t \leq x \leq a_\rho + t \mid A_x(\pi') = L_x(\pi')\}$$

and we conclude that $l_+ = l_-$, as desired. $\square$

Now set $b_{k+1} = b_\rho + t$ and $a_{k+1} = a_\rho + t$. Then π' satisfies all the assumptions of Theorem 6.1 with respect to $a_\rho, a_1, \dots, a_{k+1}, b_1, \dots, b_{k+1}$. Moreover, since

$$\sum_{\rho} \sum_{b_\rho < x \leq a_\rho} |A_{\rho|\cdot|x}(\pi)| - \sum_{\rho} \sum_{b_\rho < x \leq a_\rho} |A_{\rho|\cdot|x}(\pi')| = a_\rho - b_\rho > 0,$$

we may apply the inductive hypothesis to π' . Hence we can write $\pi' = \pi(\mathcal{E}')$. Moreover, we may assume that

$$([a_\rho + t, b_\rho + t]_\rho, l, \eta) \in \mathcal{E}'$$

for some $\eta \in \{\pm 1\}$, where $l = l_+ = l_-$ is defined in Lemma 6.4. By construction, if we set,

$$\mathcal{E} = (\mathcal{E}' \setminus \{([a_\rho + t, b_\rho + t]_\rho, l, \eta)\}) \cup \{([a_\rho, b_\rho]_\rho, l, \eta)\},$$

then $\pi = \pi(\mathcal{E})$. Therefore, we conclude that π is of Arthur type. This completes the proof of Theorem 6.1.

7. PROOF OF THEOREM 4.7

In this section, we prove the main theorems stated in Section 4. Recall that it remains to prove Theorem 4.7. Roughly speaking, we want to prove that having SPS implies being of Arthur type. To do so, we introduce a condition, Condition 7.4, which is implied by having SPS and which seems natural in the proof that a representation is of Arthur type. To motivate this condition, we first describe some attempts that led us to its formulation. We then discuss some consequences of the condition, and in the last subsection we prove the theorem.

7.1. An adapted condition. The statements of Theorems 4.7 and 5.4 have a similar form, since both use the SZ-decomposition. However, in order to prove our goal (Theorem 4.7), Theorem 5.4 is not especially well suited.

Indeed, let $\sigma \in \text{Irr}_{\text{good}}^G$ be of Arthur type. If one wants to use Theorem 5.4 to prove that

$$\text{soc}(Z_\rho[B+1, A+1]^k \rtimes \sigma)$$

is of Arthur type, one needs to verify the following condition:

- ($\star$): There exists an extended multisegment $\mathcal{E}$ with $\sigma \cong \pi(\mathcal{E})$ such that the largest k elements of $\mathcal{E}$ are of the form $([A, B]_\rho, *, *)$.

However, the hypothesis of Theorem 4.7 is that π has SPS, and σ appears in the SZ-decomposition of π . This seems quite far from ($\star$).

In this section, we would like to give a sufficient condition implying ($\star$) without using extended multisegments.

In the course of studying this problem, we considered several possible reformulations of condition ($\star$), already in the case $k = 1$. These attempts may help clarify why the final

statement takes the form it does (see Condition 7.4 below). Thus, for the remainder of this discussion, let us restrict to the case $k = 1$.

In this case, condition $(\star)$ implies that

$$\psi_{\mathcal{E}} \supset \rho \boxtimes S_{A+B+1} \boxtimes S_{A-B+1}.$$

A natural first guess is therefore that $(\star)$ might be equivalent to the irreducibility of the unitary induced representation

$$\mathrm{Sp}(\rho, a, b) \rtimes \sigma,$$

where $a = A + B + 1$ and $b = A - B + 1$. This condition might be easier to check in view of the hypotheses of Theorem 4.7. It is motivated by the tempered case with $b = 1$, that is, $A = B$. If σ is tempered, then the L -parameter ϕ_{σ} of σ contains $\rho \boxtimes S_a$ if and only if the tempered induction

$$\mathrm{Sp}(\rho, a, 1) \rtimes \sigma = \Delta_{\rho}[\frac{a-1}{2}, -\frac{a-1}{2}] \rtimes \sigma$$

is irreducible. This follows from [3, Theorem 2.2.1]. More generally, [6, Theorem 4.4] shows that $(\star)$ implies the irreducibility of

$$\mathrm{Sp}(\rho, a, b) \rtimes \sigma.$$

However, the converse is false. A counterexample was given by Mœglin in her unpublished paper.

Example 7.1 (Mœglin). *Set $\rho = \mathbf{1}_{\mathrm{GL}_1(F)}$, and we drop ρ from the notations. Consider*

$$\sigma = L(\Delta[0, -5], \Delta[1, -7], \Delta[2, -6], \Delta[3, -5]; \pi(1^-, 2^+, 3^-, 4^+, 4^+)) \in \mathrm{Irr}(\mathrm{Sp}_{98}(F)).$$

Here, $\pi(a_1^{\varepsilon_1}, \dots, a_t^{\varepsilon_t})$ means $\pi(\phi, \varepsilon)$ with $\phi = S_{2a_1+1} \oplus \dots \oplus S_{2a_t+1}$ and $\varepsilon(S_{2a_i+1}) = \varepsilon_i$. Then there are exactly two extended multisegments $\mathcal{E}$ such that $\sigma \cong \pi(\mathcal{E})$. These are

$$\mathcal{E}_1 = \left(\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \triangleleft & \ominus & \oplus & \ominus & \oplus & \triangleright & & \\ & \triangleleft & \triangleleft & \triangleleft & \oplus & \triangleright & \triangleright & \triangleright \end{array} \right)_{\rho}$$

and

$$\mathcal{E}_2 = \left(\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \triangleleft & \ominus & \oplus & \ominus & \oplus & \ominus & \oplus & \triangleright \\ & \triangleleft & \oplus & \ominus & \oplus & \triangleright & & \end{array} \right)_{\rho}.$$

*In particular, there is no $\mathcal{E}$ with $\pi(\mathcal{E}) \cong \sigma$ such that $\mathcal{E} \ni ([6, 0]_{\rho}, *, *)$ or $\mathcal{E} \ni ([6, 1]_{\rho}, *, *)$. However, using [6, Theorem 4.4], one can check that $\mathrm{Sp}(\rho, 7, 7) \rtimes \sigma$ and $\mathrm{Sp}(\rho, 8, 6) \rtimes \sigma$ are both irreducible.*

Remark 7.2. Let $\sigma \in \mathrm{Irr}_{\mathrm{good}}^G$ be of Arthur type. If $\rho \in \mathcal{C}_{\mathrm{unit}}^{\mathrm{GL}}$ and (a, b) are given, by [6, Theorem 4.4], we can check the irreducibility of $\mathrm{Sp}(\rho, a, b) \rtimes \sigma$. However, we do not know how to describe

$$\{(\rho, a, b) \mid \mathrm{Sp}(\rho, a, b) \rtimes \sigma \text{ is irreducible}\}$$

explicitly.

Our second attempt was motivated by Tadić's external approach to the classification of unitary representations of $\mathrm{GL}_n(F)$ ([45]). The main idea in the proof of the surjectivity of his classification is that, if a representation σ is not of the appropriate form, then there exists a natural unitary representation τ such that $\tau \times \sigma$ is not semisimple; in particular, its socle is different from its cosocle.

Suppose for simplicity $k = 1$ and assume

$$\pi = \mathrm{soc}(Z_\rho[B+1, A+1] \rtimes \sigma)$$

has SPS. It follows that the unitary induction

$$\mathrm{Sp}(\rho, a, b) \rtimes \pi = \left(\begin{array}{ccc} B & \dots & A \\ \vdots & \ddots & \vdots \\ -A & \dots & -B \end{array} \right)_\rho \rtimes \mathrm{soc}(Z_\rho[B+1, A+1] \rtimes \sigma),$$

where $a = A + B + 1$ and $b = A - B + 1$, is semisimple. By Tadić's formula (Corollary 2.2), we have

$$\begin{aligned} [\mathrm{Jac}(\mathrm{Sp}(\rho, a, b) \rtimes \pi)] &\geq [(Z_\rho[B+1, A+1] \times \mathrm{Sp}(\rho, a, b)) \otimes \sigma] \\ &\geq [\mathrm{Sp}(\rho, a+1, b) | \cdot |^{\frac{1}{2}} \otimes \sigma]. \end{aligned}$$

Let π' be an irreducible summand of $\mathrm{Sp}(\rho, a, b) \rtimes \pi$ such that

$$[\mathrm{Jac}(\pi')] \geq [\mathrm{Sp}(\rho, a+1, b) | \cdot |^{\frac{1}{2}} \otimes \sigma].$$

By [32, Corollary 2.8], there is a surjection

$$\mathrm{Sp}(\rho, a, b) \times Z_\rho[B+1, A+1] \twoheadrightarrow \mathrm{Sp}(\rho, a+1, b) | \cdot |^{\frac{1}{2}}$$

with an irreducible kernel. If we further assume, for example, that σ is supercuspidal, then we see that the composition

$$\pi' \hookrightarrow \mathrm{Sp}(\rho, a, b) \rtimes \pi \hookrightarrow \mathrm{Sp}(\rho, a, b) \times Z_\rho[B+1, A+1] \rtimes \sigma \twoheadrightarrow \mathrm{Sp}(\rho, a+1, b) | \cdot |^{\frac{1}{2}} \rtimes \sigma$$

is nonzero.

We conclude that (in the case $k = 1$) being SPS (together with a certain derivative assumption on σ) implies the following condition:

($\star'$): The representations $\mathrm{Sp}(\rho, a+1, b) | \cdot |^{\frac{1}{2}} \rtimes \sigma$ and $\mathrm{Sp}(\rho, a, b) \rtimes \mathrm{soc}(Z_\rho[B+1, A+1] \rtimes \sigma)$ have a common irreducible subrepresentation, where $a = A + B + 1$ and $b = A - B + 1$.

This is a strong condition, and it may be close to being equivalent to ($\star$). The general statement is as follows.

Proposition 7.3. *Let $\pi(\phi, \varepsilon) \in \mathrm{Irr}^G$, and let*

$$(\tau_{\mathrm{bad}}^-, \{\tau_j^-\}_{1 \leq j \leq s}, \{\tau_j^+\}_{1 \leq j \leq r}, \{\pi_j\}_{0 \leq j \leq r+s})$$

be its SZ-decomposition. Assume that $\pi(\phi, \varepsilon)$ has SPS. Let S be a product of unitary Speh representations.

- (1) *Assume $s > 0$ and fix $r \leq j < r + s$. Then there exists an irreducible representation π'_{j+1} such that:*
 - $\pi'_{j+1} \hookrightarrow S \rtimes \pi_{j+1}$;
 - $\pi'_{j+1} \hookrightarrow \mathrm{soc}(\tau_{j-r+1}^- \times S) \rtimes \pi_j$.

(2) Assume $r > 0$ and fix $0 \leq j < r$. Then there exists an irreducible representation π'_{j+1} such that:

- $\pi'_{j+1} \hookrightarrow S \rtimes \pi_{j+1}$;
- $\pi'_{j+1} \hookrightarrow \mathrm{soc}(\tau_{j+1}^+ \times S) \rtimes \pi_j$.

Proof. We treat only part (2), since the proof of part (1) is similar. In fact, the argument for (2) essentially subsumes the argument for (1).

We fix $0 \leq j < r$. By Remark 3.3, we have the inclusions

$$\begin{aligned}
 \pi(\phi, \varepsilon) &\hookrightarrow \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s} \\
 &\hookrightarrow \tau_s^- \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s-1} \\
 &\quad \vdots \\
 &\hookrightarrow \tau_s^- \times \cdots \times \tau_1^- \times \tau_{\mathrm{bad}}^- \rtimes \pi_r \\
 &\hookrightarrow \tau_s^- \times \cdots \times \tau_1^- \times \tau_r^+ \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r-1} \\
 &\quad \vdots \\
 &\hookrightarrow \tau_s^- \times \cdots \times \tau_1^- \times \tau_r^+ \times \cdots \times \tau_1^+ \times \tau_{\mathrm{bad}}^- \rtimes \pi_0.
 \end{aligned}$$

By Geometric Lemma (Theorem 2.1), we have

$$[\mathrm{Jac}(S \rtimes \pi(\phi, \varepsilon))] \geq \tau_s^- \otimes \cdots \otimes \tau_1^- \otimes \tau_r^+ \otimes \cdots \otimes \tau_1^+ \otimes S \otimes \tau_{\mathrm{bad}}^- \otimes \pi_0.$$

Therefore, there exists an irreducible subquotient π''_{r+s} of $S \rtimes \pi(\phi, \varepsilon)$ such that

$$(*) \quad [\mathrm{Jac}(\pi''_{r+s})] \geq \tau_s^- \otimes \cdots \otimes \tau_1^- \otimes \tau_r^+ \otimes \cdots \otimes \tau_1^+ \otimes S \otimes \tau_{\mathrm{bad}}^- \otimes \pi_0.$$

Since $\pi(\phi, \varepsilon)$ has SPS, the induction $S \rtimes \pi(\phi, \varepsilon)$ is semisimple. Therefore, $\pi''_{r+s} \hookrightarrow S \rtimes \pi(\phi, \varepsilon)$.

By [30, Corollary 4.9], we know that $\tau_s^- \times S$ is SI. We have

$$\begin{aligned}
 [\pi''_{r+s}] &\leq [S \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s}] \\
 &\leq [S \times \tau_s^- \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s-1}] \\
 &= [\tau_s^- \times S \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s-1}] \\
 &= [\mathrm{soc}(\tau_s^- \times S) \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s-1}] + \left[\frac{\tau_s^- \times S}{\mathrm{soc}(\tau_s^- \times S)} \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s-1} \right].
 \end{aligned}$$

Recall that we denote $\tau_{s-i+1}^- = \Delta_{\rho_i}[x_i, y_i]^{k_i}$ such that $y_1 \leq \cdots \leq y_s$ and if $i < j$ and $y_i = y_j$ then $x_i \leq x_j$. By taking $D_{\rho_1|\cdot|y_1}^{(k_1)} \circ \cdots \circ D_{\rho_1|\cdot|x_1}^{(k_1)}$, we see that $[\pi''_{r+s}]$ does not appear in

$$\left[\frac{\tau_s^- \times S}{\mathrm{soc}(\tau_s^- \times S)} \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s-1} \right].$$

Namely, the composition map

$$\begin{aligned}
 \pi''_{r+s} &\hookrightarrow S \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s} \\
 &\hookrightarrow S \times \tau_s^- \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s-1} \\
 &\rightarrow \mathrm{soc}(\tau_s^- \times S) \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s-1} \\
 &\hookrightarrow \tau_s^- \times S \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r+s-1}
 \end{aligned}$$

is nonzero. In particular, there is an irreducible subquotient π''_{r+s-1} of $S \times \tau_{\text{bad}}^- \rtimes \pi_{r+s-1}$ such that

$$\pi''_{r+s} \hookrightarrow \tau_s^- \rtimes \pi''_{r+s-1}.$$

By construction, the Langlands data of π''_{r+s} are obtained by adding $\tau_s^- = \Delta_{\rho_1}[x_1, y_1]^{k_1}$ to those of π''_{r+s-1} . Thus, π''_{r+s-1} is uniquely characterized by the condition

$$[\text{Jac}(\pi''_{r+s})] = \tau_s^- \otimes (m \cdot \pi''_{r+s-1}) + \sum_{\substack{\tau \in \text{Irr}(\text{GL}_{n_s}(F)) \\ \tau \not\cong \tau_s}} \tau \otimes [\Pi_\tau],$$

for some representation Π_τ and some multiplicity $m > 0$, where n_s is such that $\tau_s^- \in \text{Irr}(\text{GL}_{n_s}(F))$. By applying Frobenius reciprocity to the inclusion map $\pi''_{r+s} \hookrightarrow \tau_s^- \times S \times \tau_{\text{bad}}^- \rtimes \pi_{r+s-1}$, we have a nonzero equivariant map

$$\tau_s^- \boxtimes \pi''_{r+s-1} \rightarrow \tau_s^- \boxtimes (S \times \tau_{\text{bad}}^- \rtimes \pi_{r+s-1}).$$

Since π''_{r+s-1} is irreducible, we obtain an inclusion $\pi''_{r+s-1} \hookrightarrow S \times \tau_{\text{bad}}^- \rtimes \pi_{r+s-1}$. On the other hand, by (*), we see that

$$[\text{Jac}(\pi''_{r+s-1})] \geq \tau_{s-1}^- \otimes \cdots \otimes \tau_1^- \otimes \tau_r^+ \otimes \cdots \otimes \tau_1^+ \otimes S \otimes \tau_{\text{bad}}^- \otimes \pi_0.$$

Repeating this argument, for $0 \leq i \leq s$, there is an irreducible representation π''_{r+s-i} such that

- $\pi''_{r+s-i} \hookrightarrow S \times \tau_{\text{bad}}^- \rtimes \pi_{r+s-i}$; and
- $[\text{Jac}(\pi''_{r+s-i})] \geq \tau_{s-i}^- \otimes \cdots \otimes \tau_1^- \otimes \tau_r^+ \otimes \cdots \otimes \tau_1^+ \otimes S \otimes \tau_{\text{bad}}^- \otimes \pi_0$.

In particular, considering $i = s$, we have an irreducible representation π''_r such that

- $\pi''_r \hookrightarrow S \times \tau_{\text{bad}}^- \rtimes \pi_r$; and
- $[\text{Jac}(\pi''_r)] \geq \tau_r^+ \otimes \cdots \otimes \tau_1^+ \otimes S \otimes \tau_{\text{bad}}^- \otimes \pi_0$.

By the same argument, we have

$$\begin{aligned} [\pi''_r] &\leq [S \times \tau_{\text{bad}}^- \rtimes \pi_r] \\ &\leq [S \times \tau_r^+ \times \tau_{\text{bad}}^- \rtimes \pi_{r-1}] \\ &= [\tau_r^+ \times S \times \tau_{\text{bad}}^- \rtimes \pi_{r-1}] \\ &= [\text{soc}(\tau_r^+ \times S) \times \tau_{\text{bad}}^- \rtimes \pi_{r-1}] + \left[\frac{\tau_r^+ \times S}{\text{soc}(\tau_r^+ \times S)} \times \tau_{\text{bad}}^- \rtimes \pi_{r-1} \right]. \end{aligned}$$

Recall that $\tau_{r-i+1}^+ = Z_{\rho'_i}[-x'_i, -y'_i]^{k'_i}$ such that $y'_1 \leq \cdots \leq y'_r$ and if $i < j$ and $y'_i = y'_j$ then $x'_i \leq x'_j$. Applying $D_{\rho'_1|\cdot|^{-y'_1}}^{(k'_1)} \circ \cdots \circ D_{\rho'_1|\cdot|^{-x'_1}}^{(k'_1)}$, we see that $[\pi''_r]$ does not occur in

$$\left[\frac{\tau_r^+ \times S}{\text{soc}(\tau_r^+ \times S)} \times \tau_{\text{bad}}^- \rtimes \pi_{r-1} \right].$$

Namely, the composition map

$$\begin{aligned} \pi''_r &\hookrightarrow S \times \tau_{\text{bad}}^- \rtimes \pi_r \\ &\hookrightarrow S \times \tau_r^+ \times \tau_{\text{bad}}^- \rtimes \pi_{r-1} \\ &\twoheadrightarrow \text{soc}(\tau_r^+ \times S) \times \tau_{\text{bad}}^- \rtimes \pi_{r-1} \\ &\hookrightarrow \tau_r^+ \times S \times \tau_{\text{bad}}^- \rtimes \pi_{r-1} \end{aligned}$$

is nonzero, and we can find an irreducible subquotient π''_{r-1} of $S \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r-1}$ such that

$$\pi''_r \hookrightarrow \tau_r^+ \rtimes \pi''_{r-1}.$$

By applying Frobenius reciprocity to $\pi''_r \hookrightarrow \tau_r^+ \times S \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r-1}$, we obtain an inclusion $\pi''_{r-1} \hookrightarrow S \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r-1}$. Moreover, π''_{r-1} satisfies that

$$[\mathrm{Jac}(\pi''_{r-1})] \geq \tau_{r-1}^+ \otimes \cdots \otimes \tau_1^+ \otimes S \otimes \tau_{\mathrm{bad}}^- \otimes \pi_0.$$

Repeating this argument for $0 \leq i \leq r-j-1$, we obtain an irreducible representation π''_{r-i} such that

- $\pi''_{r-i} \hookrightarrow S \times \tau_{\mathrm{bad}}^- \rtimes \pi_{r-i}$; and
- $[\mathrm{Jac}(\pi''_{r-i})] \geq \tau_{r-i}^+ \otimes \cdots \otimes \tau_1^+ \otimes S \otimes \tau_{\mathrm{bad}}^- \otimes \pi_0$.

Moreover, by the argument for $i = r-j-1$, the composition

$$\pi''_{j+1} \hookrightarrow S \times \tau_{\mathrm{bad}}^- \rtimes \pi_{j+1} \hookrightarrow S \times \tau_{j+1}^+ \times \tau_{\mathrm{bad}}^- \rtimes \pi_j \twoheadrightarrow \mathrm{soc}(\tau_{j+1}^+ \times S) \times \tau_{\mathrm{bad}}^- \rtimes \pi_j$$

is nonzero.

Finally, let $\pi'_{j+1} \in \mathrm{Irr}_{\mathrm{good}}^G$ be such that $\pi''_{j+1} = \mathrm{soc}(\tau_{\mathrm{bad}}^- \rtimes \pi'_{j+1})$. Applying Frobenius reciprocity again to the embeddings

$$\pi''_{j+1} \hookrightarrow \tau_{\mathrm{bad}}^- \times S \rtimes \pi_{j+1} \quad \text{and} \quad \pi''_{j+1} \hookrightarrow \tau_{\mathrm{bad}}^- \times \mathrm{soc}(\tau_{j+1}^+ \times S) \rtimes \pi_j,$$

we obtain inclusions

$$\pi'_{j+1} \hookrightarrow S \rtimes \pi_{j+1} \quad \text{and} \quad \pi'_{j+1} \hookrightarrow \mathrm{soc}(\tau_{j+1}^+ \times S) \rtimes \pi_j.$$

This completes the proof of Proposition 7.3. $\square$

In the following subsection, we study the consequences of this common-subrepresentation property in its own right, abstracting away from the particular situation in which it was obtained. We will be particularly interested in the case where S is a product of a very explicit unitary Speh representation. We formulate this condition in the following two forms, corresponding to the two possible signs of τ .

Condition 7.4. Let $\sigma \in \mathrm{Irr}_{\mathrm{good}}^G$.

- (1) Fix half-integers $B < 1$ and $A > 0$ such that $A + B \geq 0$. Set $\tau = \Delta_\rho[B-1, -A-1]^k$ and let S be the product of k copies of the Speh representation

$$\left(\begin{array}{ccc} B & \cdots & A \\ \vdots & \ddots & \vdots \\ -A & \cdots & -B \end{array} \right)_\rho,$$

where $\rho \in \mathcal{C}_{\mathrm{sd}}^{\mathrm{GL}}$. Set $\pi = \mathrm{soc}(\tau \rtimes \sigma)$. Suppose that

- $D_{\rho|\cdot|^b, \dots, \rho|\cdot|^a}(\sigma) = 0$ for any $a \leq b < 0$ such that $a < -A-1$, or $b \leq B-1$ and $a = -A-1$;
- there is $\pi' \in \mathrm{Irr}_{\mathrm{good}}^G$ such that $\pi' \hookrightarrow S \rtimes \pi$ and $\pi' \hookrightarrow \mathrm{soc}(\tau \times S) \rtimes \sigma$, i.e.,

$$S \rtimes \mathrm{soc}(\tau \rtimes \sigma) \quad \text{and} \quad \mathrm{soc}(\tau \times S) \rtimes \sigma$$

have a common irreducible subrepresentation.

- (2) Fix half-integers $A \geq B > -1$. Set $\tau = Z_\rho[B+1, A+1]^k$ and let S be the product of k copies of the Speh representation

$$\begin{pmatrix} B & \cdots & A \\ \vdots & \ddots & \vdots \\ -A & \cdots & -B \end{pmatrix}_\rho,$$

where $\rho \in \mathcal{C}_{\text{sd}}^{\text{GL}}$. Set $\pi = \text{soc}(\tau \rtimes \sigma)$. Suppose that

- $D_{\rho|\cdot|^b, \dots, \rho|\cdot|^a}(\sigma) = 0$ for any $a \geq b > 0$ such that $a > A+1$, or $b \geq B+1$ and $a = A+1$;
- there is $\pi' \in \text{Irr}_{\text{good}}^G$ such that $\pi' \hookrightarrow S \rtimes \pi$ and $\pi' \hookrightarrow \text{soc}(\tau \times S) \rtimes \sigma$, i.e.,

$$S \rtimes \text{soc}(\tau \rtimes \sigma) \quad \text{and} \quad \text{soc}(\tau \times S) \rtimes \sigma$$

have a common irreducible subrepresentation.

7.2. Inequalities for Jacquet modules. In this subsection, we establish several inequalities for Jacquet modules under Condition 7.4. We retain the notation of the previous discussion.

Define a sequence $\{x_j\}_{1 \leq j \leq ab}$ with $a = A+B+1$ and $b = A-B+1$ by

$$\text{Sp}(\rho, a, b) = \begin{pmatrix} B & \cdots & A \\ \vdots & \ddots & \vdots \\ -A & \cdots & -B \end{pmatrix}_\rho = \begin{pmatrix} x_1 & x_{a+1} & \cdots & x_{a(b-1)+1} \\ x_2 & x_{a+2} & \cdots & x_{a(b-1)+2} \\ \vdots & \vdots & \ddots & \vdots \\ x_a & x_{2a} & \cdots & x_{ab} \end{pmatrix}_\rho.$$

In particular, $-A \leq x_j \leq A$ for $1 \leq j \leq ab$. To $S = \text{Sp}(\rho, a, b)^k$, we attach an operator D_S defined by

$$D_S = D_{\rho|\cdot|^{x_{ab}}}^{(k)} \circ \cdots \circ D_{\rho|\cdot|^{x_1}}^{(k)}.$$

For a representation Π , we call $D_S(\Pi)$ the S -derivative of Π .

Using the operator D_S , we prove the following lemma. This is the most technical part of the proof of Theorem 4.7. It appears naturally in Mœglin's explicit construction of A -packets (see Section 5.1).

Lemma 7.5. Let $\sigma \in \text{Irr}_{\text{good}}^G$.

- (1) Assume Condition 7.4 (1), and let

$$\pi = \text{soc}(\tau \rtimes \sigma).$$

Fix integers $t_1, \dots, t_k > A+B$, and let

$$S' = \bigotimes_{i=1}^k \begin{pmatrix} B-t_i & \cdots & B-2 \\ \vdots & \ddots & \vdots \\ -A-t_i & \cdots & -A-2 \end{pmatrix}_\rho.$$

Then

$$D_S(\text{soc}(\text{soc}(S' \times \tau \times S) \rtimes \sigma)) \geq \text{soc}(S' \rtimes \pi).$$

- (2) Assume Condition 7.4 (2), and let

$$\pi = \text{soc}(\tau \rtimes \sigma).$$

Fix integers $t_1, \dots, t_k > A - B$, and set

$$S' = \bigotimes_{i=1}^k \begin{pmatrix} B + t_i & \dots & A + t_i \\ \vdots & \ddots & \vdots \\ B + 2 & \dots & A + 2 \end{pmatrix}_\rho.$$

Then

$$D_S(\mathrm{soc}(\mathrm{soc}(S' \times \tau \times S) \rtimes \sigma)) \geq \mathrm{soc}(S' \rtimes \pi).$$

Observe that $\mathrm{soc}(S' \rtimes \pi)$ is irreducible by Property (iii) in Section 2.3. Moreover, assertions (1) and (2) of Lemma 7.5 (and likewise of Condition 7.4) are equivalent under Aubert duality. For the remainder of this subsection, we work in the setting of (1).

First, we show that there is $\pi_j \in \mathrm{Irr}_{\mathrm{good}}^G$ for $0 \leq j \leq ab$ such that

- $\pi_0 = \pi'$ and $\pi_{ab} = \pi$;
- for $0 \leq j \leq ab$, we have $\pi_j \hookrightarrow L_j \rtimes \pi$, where $L_0 = S$ and, for $1 \leq j \leq ab$,

$$L_j = D_{\rho|\cdot|^{x_j}}^{(k)} \circ \dots \circ D_{\rho|\cdot|^{x_1}}^{(k)}(S) \in \mathrm{Irr}^{\mathrm{GL}};$$

- $\pi_j \leq D_{\rho|\cdot|^{x_j}}^{(k)}(\pi_{j-1})$ for $1 \leq j \leq ab$.

Indeed, by induction, we assume the existence of π_{j-1} . Then

$$\pi_{j-1} \hookrightarrow L_{j-1} \rtimes \pi \hookrightarrow (\rho|\cdot|^{x_j})^k \times L_j \rtimes \pi.$$

By Frobenius reciprocity, we have a nonzero map

$$\mathrm{Jac}(\pi_{j-1}) \rightarrow (\rho|\cdot|^{x_j})^k \boxtimes (L_j \rtimes \pi).$$

Hence there exists an irreducible subrepresentation π_j of $L_j \rtimes \pi$ such that $(\rho|\cdot|^{x_j})^k \boxtimes \pi_j$ is contained in the image of this map. Therefore,

$$D_{\rho|\cdot|^{x_j}}^{(k)}(\pi_{j-1}) \geq \pi_j.$$

In particular, when $j = ab$, since $L_{ab} = \mathbf{1}_{\mathrm{GL}_0(F)}$, we have $\pi_{ab} \hookrightarrow \pi$, and hence $\pi_{ab} = \pi$. Since π_j need not be unique, we fix one such irreducible subrepresentation π_j for each j .

Lemma 7.6. *Let $\sigma \in \mathrm{Irr}_{\mathrm{good}}^G$ and $\pi = \mathrm{soc}(\tau \rtimes \sigma)$ be as in Lemma 7.5 (1). For $0 \leq j \leq ab$, fix $\pi_j \in \mathrm{Irr}_{\mathrm{good}}^G$ as above. Then there exists a unique $\sigma_j \in \mathrm{Irr}_{\mathrm{good}}^G$ such that $\pi_j = \mathrm{soc}(\tau \rtimes \sigma_j)$. In particular, $\sigma_{ab} = \sigma$. Moreover, for $1 \leq j \leq ab$,*

- $\sigma_j \leq [L_j \rtimes \sigma]$;
- for any $a \leq -A - 1 \leq b < B - 1$, we have $D_{\rho|\cdot|^a} \circ \dots \circ D_{\rho|\cdot|^b} \circ D_{\rho|\cdot|^{x_j}}^{(k)}(\sigma_{j-1}) = 0$;
- $\sigma_j \leq D_{\rho|\cdot|^{x_j}}^{(k)}(\sigma_{j-1})$.

Proof. We first prove uniqueness and the asserted properties, assuming the existence of such a σ_j . We prove existence at the end. Note that $\tau \rtimes \sigma_j$ is SI for any $\sigma_j \in \mathrm{Irr}^G$ by Property (iii) in Section 2.3. Moreover, if $\pi_j = \mathrm{soc}(\tau \rtimes \sigma_j)$, and if we let $k_{B-1}, \dots, k_{-A-1} \geq k$ be the integers such that

$$D_{\rho|\cdot|^{B-i}}^{(k_{B-i})} \circ \dots \circ D_{\rho|\cdot|^{B-1}}^{(k_{B-1})}(\pi_j) \neq 0$$

and k_{B-i} is maximal in this condition for $i = 1, \dots, A + B + 1$, then since $B - 1 < 0$, we see that $D_{\rho|\cdot|^{-A-1}}^{(k_{-A-1})} \circ \dots \circ D_{\rho|\cdot|^{B-1}}^{(k_{B-1})}(\pi_j)$ is irreducible and σ_j is uniquely determined by

$$D_{\rho|\cdot|^{-A-1}}^{(k_{-A-1})} \circ \dots \circ D_{\rho|\cdot|^{B-1}}^{(k_{B-1})}(\pi_j) \cong D_{\rho|\cdot|^{-A-1}}^{(k_{-A-1}-k)} \circ \dots \circ D_{\rho|\cdot|^{B-1}}^{(k_{B-1}-k)}(\sigma_j).$$

In particular, we must have $\sigma_{ab} = \sigma$.

Write $\pi_j = L(\mathbf{m}_j; \pi(\phi_j, \varepsilon_j))$ as in the Langlands classification. First, we show that $\mathbf{m}_j$ does not contain $[x, y]_\rho$ such that $x < B - 1$ and $y < -A$. Indeed, if there were to exist such a segment in $\mathbf{m}_j$, then we could take $C < B - 1$ such that $D_{\rho|\cdot|^{-A-1}} \circ \dots \circ D_{\rho|\cdot|^C}(\pi_j) \neq 0$. However, since

$$\pi_j \hookrightarrow L_j \rtimes \pi$$

and since the cuspidal support of $\mathrm{Sp}(\rho, a, b)$, and hence the one of L_j , does not contain $\rho|\cdot|^{\pm(A+1)}$, we see that there is $-A - 1 \leq z < B - 1$ such that $D_{\rho|\cdot|^z}(\pi) \neq 0$. This contradicts that $\pi = \mathrm{soc}(\tau \rtimes \sigma)$ and $D_{\rho|\cdot|^z}(\sigma) = 0$ for $z < B - 1$. Therefore, to show the existence of $\sigma_j \in \mathrm{Irr}_{\mathrm{good}}^G$ such that $\pi_j = \mathrm{soc}(\tau \rtimes \sigma_j)$, by the Langlands classification, it is enough to prove that $[\mathrm{Jac}(\pi_j)] \geq \tau \otimes \sigma_j$ for some $\sigma_j \neq 0$. Moreover, if we were to know the existence of σ_j such that $\pi_j = \mathrm{soc}(\tau \rtimes \sigma_j)$, then $[\mathrm{Jac}(\pi_j)]$ would be of the form

$$[\mathrm{Jac}(\pi_j)] = \tau \otimes \sigma_j + \sum_{\substack{\tau' \in \mathrm{Irr}^{\mathrm{GL}} \\ \tau' \neq \tau}} \tau' \otimes [\Pi_{\tau'}].$$

See [10, Proposition 3.7]. Applying an appropriate Jacquet functor to the inclusion $\pi_j \hookrightarrow L_j \times \tau \rtimes \sigma$, we see that $\sigma_j \leq [L_j \rtimes \sigma]$.

To prove the second claim, suppose for a contradiction that

$$D_{\rho|\cdot|^a} \circ \dots \circ D_{\rho|\cdot|^b} \circ D_{\rho|\cdot|^{x_j}}^{(k)}(\sigma_{j-1}) \neq 0$$

for some $a \leq -A - 1 \leq b < B - 1$. Since

$$\sigma_{j-1} \leq [L_{j-1} \rtimes \sigma],$$

and $\rho|\cdot|^a$ is not in the cuspidal support of L_{j-1} , Tadić's formula (Corollary 2.2) implies that there exists a subsequence $(y_1, \dots, y_t)$ of $(a, a + 1, \dots, b)$, with $y_1 = a$, and an integer $0 \leq k' \leq k$, such that

$$D_{\rho|\cdot|^{y_1}} \circ \dots \circ D_{\rho|\cdot|^{y_t}} \circ D_{\rho|\cdot|^{x_j}}^{(k')}(\sigma) \neq 0.$$

If $(y_1, \dots, y_t, x_j) = (a, a + 1, \dots, B - 1)$, it contradicts the first assumption in Condition 7.4 (1). But, if they are different, by the remark at the end of Subsection 2.1, we would find $c < B - 1$ with $c \leq a$ such that $D_{\rho|\cdot|^c, \dots, \rho|\cdot|^a}(\sigma) \neq 0$, which again contradicts the first assumption in Condition 7.4 (1). This proves the second assertion.

Moreover, since

$$\pi_j = \mathrm{soc}(\tau \rtimes \sigma_j) \leq D_{\rho|\cdot|^{x_j}}^{(k)}(\pi_{j-1}) \leq D_{\rho|\cdot|^{x_j}}^{(k)}(\tau \rtimes \sigma_{j-1}),$$

we would have

$$\sigma_j \leq \left(D_{\rho|\cdot|^{-A-1}}^{(k)} \circ \dots \circ D_{\rho|\cdot|^{B-1}}^{(k)} \right) \circ D_{\rho|\cdot|^{x_j}}^{(k)}(\tau \rtimes \sigma_{j-1}).$$

By Tadić's formula (Corollary 2.2), it would imply that $\sigma_j \leq D_{\rho|\cdot|^{x_j}}^{(k)}(\sigma_{j-1})$.

Now we prove the existence of σ_j . To do this, we prepare some notations. Fix $0 \leq j \leq ab$. Let

- $d, m \geq 0$ be such that $S \in \mathrm{Irr}(\mathrm{GL}_d(F))$ and $L_j \in \mathrm{Irr}(\mathrm{GL}_m(F))$;
- G_S, G_L and G be the classical groups such that $S \rtimes \pi \in \mathrm{Rep}(G_S)$, $L_j \rtimes \pi \in \mathrm{Rep}(G_L)$ and $L_j \times S \rtimes \pi \in \mathrm{Rep}(G)$;
- $P = MN_P$ be the standard parabolic subgroup of G with $M \cong \mathrm{GL}_m(F) \times G_S$;
- $\iota: \mathrm{GL}_m(F) \times G_S \xrightarrow{\sim} M$ be a fixed isomorphism.

There is a canonical surjection $L_j \times S \rtimes \pi \twoheadrightarrow L_j \boxtimes (S \rtimes \pi)$ which is given by

$$L_j \times S \rtimes \pi \cong L_j \rtimes (S \rtimes \pi) \twoheadrightarrow L_j \boxtimes (S \rtimes \pi).$$

Here, $f \in L_j \rtimes (S \rtimes \pi)$ is regarded as a two variable smooth function

$$f: G \times G_S \rightarrow L_j \boxtimes S \boxtimes \pi$$

such that

$$f(n \cdot \iota(a, h) \cdot g, h') = |\det(a)|^\delta L_j(a) f(g, h' h)$$

for $n \in N_P, a \in \mathrm{GL}_m(F), h, h' \in G_S$ and $g \in G$, where $\delta \in \mathbb{R}$ is independent of f . Then the isomorphism $L_j \rtimes (S \rtimes \pi) \xrightarrow{\sim} L_j \times S \rtimes \pi$ is given by $f(g, h) \mapsto f(g, \mathbf{1})$, whereas, the surjection $L_j \rtimes (S \rtimes \pi) \twoheadrightarrow L_j \boxtimes (S \rtimes \pi)$ is given by $f(g, h) \mapsto f(\mathbf{1}, h)$. In particular, the surjection $L_j \times S \rtimes \pi \twoheadrightarrow L_j \boxtimes (S \rtimes \pi)$ is described by $f(g, \mathbf{1}) \mapsto f(\iota(\mathbf{1}, h), \mathbf{1})$. This surjection factors through the M -equivariant map

$$\mathrm{Jac}_P(L_j \times S \rtimes \pi) \twoheadrightarrow L_j \boxtimes (S \rtimes \pi),$$

which we denote by Res_{G_S} . Similarly, we have an evaluation map $\mathrm{ev}: \mathrm{Jac}(L_j \rtimes \pi) \twoheadrightarrow L_j \boxtimes \pi$ at $\mathbf{1}$. This gives a surjection

$$(\mathbf{1}_{\mathrm{GL}_0(F)} \boxtimes S) \rtimes \mathrm{Jac}(L_j \rtimes \pi) \twoheadrightarrow (\mathbf{1}_{\mathrm{GL}_0(F)} \boxtimes S) \rtimes (L_j \boxtimes \pi) \cong L_j \boxtimes (S \rtimes \pi),$$

which is also denoted by ev . If we regard the left-hand (resp. right-hand) side as a space of smooth functions on $\mathrm{GL}_m(F) \times G_S \times G_L$ (resp. G_S), this map is just the evaluation at identity elements of $\mathrm{GL}_m(F)$ and G_L . Finally, by applying the contragredient and the MVW functors to the injection $\pi' \hookrightarrow S \rtimes \pi$, we obtain, by [8, Lemma 2.2], a surjection $S \rtimes \pi \twoheadrightarrow \pi'$.

Since $S \times L_j$ is irreducible by [30, Proposition 6.6], we have $S \times L_j \rtimes \pi \cong L_j \times S \rtimes \pi$. There is an inclusion map $S \boxtimes \pi_j \hookrightarrow S \boxtimes (L_j \rtimes \pi)$ of representations of $\mathrm{GL}_d(F) \times G_L$, which is regarded as a Levi subgroup of G . It gives an inclusion

$$\mathrm{Jac}_P(S \rtimes \pi_j) \hookrightarrow \mathrm{Jac}_P(S \rtimes (L_j \rtimes \pi)).$$

By the functoriality of Geometric Lemma (Theorem 2.1), we have subspaces $F(S \boxtimes \pi_j) \subset S \rtimes \pi_j$ and $F(S \boxtimes (L_j \rtimes \pi)) \subset S \rtimes (L_j \rtimes \pi)$ such that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Jac}_P(F(S \boxtimes \pi_j)) & \hookrightarrow & \mathrm{Jac}_P(F(S \boxtimes (L_j \rtimes \pi))) \\ \downarrow & & \downarrow \\ (\mathbf{1}_{\mathrm{GL}_0(F)} \boxtimes S) \rtimes \mathrm{Jac}(\pi_j) & \hookrightarrow & (\mathbf{1}_{\mathrm{GL}_0(F)} \boxtimes S) \rtimes \mathrm{Jac}(L_j \rtimes \pi). \end{array}$$

Incorporating the maps obtained in the previous paragraph into the picture, we obtain the following diagram:

$$\begin{array}{ccccc}
\mathrm{Jac}_P(S \rtimes \pi_j) & \hookrightarrow & \mathrm{Jac}_P(S \times L_j \rtimes \pi) & \xlongequal{\quad} & \mathrm{Jac}_P(L_j \times S \rtimes \pi) \\
\cup & & \cup & & \downarrow \mathrm{Res}_{G_S} \\
\mathrm{Jac}_P(F(S \boxtimes \pi_j)) & \hookrightarrow & \mathrm{Jac}_P(F(S \boxtimes (L_j \rtimes \pi))) & & L_j \boxtimes (S \rtimes \pi) \\
\downarrow & & \downarrow & \nearrow \mathrm{ev} & \downarrow \\
(\mathbf{1}_{\mathrm{GL}_0(F)} \boxtimes S) \rtimes \mathrm{Jac}(\pi_j) & \hookrightarrow & (\mathbf{1}_{\mathrm{GL}_0(F)} \boxtimes S) \rtimes \mathrm{Jac}(L_j \rtimes \pi) & & L_j \boxtimes \pi'
\end{array}$$

We claim that this diagram is commutative. In fact, the isomorphism $S \times L_j \rtimes \pi \xrightarrow{\sim} L_j \times S \rtimes \pi$ is given by the meromorphic continuation of the Jacquet integral

$$Jf(g) = \int_{\tilde{N}_P} f(w^{-1}ug)du,$$

where $w \in G$ is a representative of a certain Weyl element, and $\tilde{N}_P$ is a quotient of N_P . In particular, the composition with Res_{G_S} is induced by $f \mapsto Jf(\iota(\mathbf{1}, h))$. On the other hand, the subspace $F(S \boxtimes (L_j \rtimes \pi))$ of $S \times L_j \rtimes \pi$ is taken as a subspace consisting of f such that $Jf(\iota(a, h))$ converges absolutely for $(a, h) \in \mathrm{GL}_m(F) \times G_S$, and this is the image of f under the surjection $F(S \boxtimes (L_j \rtimes \pi)) \twoheadrightarrow (\mathbf{1}_{\mathrm{GL}_0(F)} \boxtimes S) \rtimes \mathrm{Jac}(L_j \rtimes \pi)$. Hence the composition with ev is given by $f \mapsto Jf(\iota(\mathbf{1}, h))$. This shows the commutativity of the right square.

Note that the composition

$$(\mathbf{1}_{\mathrm{GL}_0(F)} \boxtimes S) \rtimes \mathrm{Jac}(\pi_j) \hookrightarrow (\mathbf{1}_{\mathrm{GL}_0(F)} \boxtimes S) \rtimes \mathrm{Jac}(L_j \rtimes \pi) \xrightarrow{\mathrm{ev}} L_j \boxtimes (S \rtimes \pi)$$

is surjective since it is induced from a surjection $\mathrm{Jac}(\pi_j) \twoheadrightarrow L_j \boxtimes \pi$. Hence the above commutative diagram shows that we have a surjection $\mathrm{Jac}_P(S \rtimes \pi_j) \twoheadrightarrow L_j \boxtimes \pi'$, which implies, by Frobenius reciprocity, a nonzero map

$$S \rtimes \pi_j \rightarrow L_j \rtimes \pi'.$$

Since $L_j \times \mathrm{soc}(\tau \times S)$ is irreducible by [30, Proposition 6.6], we have

$$\begin{aligned}
S \rtimes \pi_j &\rightarrow L_j \rtimes \pi' \\
&\hookrightarrow L_j \times \mathrm{soc}(\tau \times S) \rtimes \sigma \\
&\cong \mathrm{soc}(\tau \times S) \times L_j \rtimes \sigma \\
&\hookrightarrow \tau \times S \times L_j \rtimes \sigma.
\end{aligned}$$

This implies $[\mathrm{Jac}(S \rtimes \pi_j)] \geq \tau \otimes S \otimes L_j \otimes \sigma \neq 0$. Since $\rho|\cdot|^{-A-1} \notin \mathrm{supp}(S)$, by Tadić's formula (Corollary 2.2), there is an irreducible representation $\tau'' \otimes \sigma'_j$ with $[\mathrm{Jac}(\pi_j)] \geq \tau'' \otimes \sigma'_j$ such that

$$\underbrace{\{\rho|\cdot|^{-A-1}, \dots, \rho|\cdot|^{-A-1}\}}_k \subset \mathrm{supp}(\tau'') \subset \mathrm{supp}(\tau)$$

as multisets. By the condition on $\mathfrak{m}_j$ we have proven earlier, we must have $\tau'' = \tau$, as desired. This completes the proof of Lemma 7.6. $\square$

Now we can prove Lemma 7.5.

Proof of Lemma 7.5. As noted above, it suffices to prove (1). Accordingly, we assume that we are in the setting of (1).

For $\sigma_j \in \mathrm{Irr}_{\mathrm{good}}^G$ defined in Lemma 7.6, we claim that

$$\left(D_{\rho|\cdot|^{x_j}}^{(k)} \circ \cdots \circ D_{\rho|\cdot|^{x_1}}^{(k)} \right) (\mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_0)) \geq \mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_j)$$

for $0 \leq j \leq ab$. This claim for $j = ab$ yields that

$$D_S(\mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_0)) \geq \mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_{ab}) = \mathrm{soc}(S' \rtimes \pi).$$

Since

$$\begin{aligned} \mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_0) &= \mathrm{soc}(S' \rtimes \mathrm{soc}(\tau \rtimes \sigma_0)) \\ &= \mathrm{soc}(S' \rtimes \pi') \leq [\mathrm{soc}(\mathrm{soc}(S' \times \tau \times S) \rtimes \sigma)], \end{aligned}$$

this implies Lemma 7.5.

The claim for $j = 0$ is trivial. Fix $1 \leq j \leq ab$, and suppose that the claim holds for $j - 1$. Then

$$\left(D_{\rho|\cdot|^{x_j}}^{(k)} \circ \cdots \circ D_{\rho|\cdot|^{x_1}}^{(k)} \right) (\mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_0)) \geq D_{\rho|\cdot|^{x_j}}^{(k)} (\mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_{j-1})).$$

We will show that $D_{\rho|\cdot|^{x_j}}^{(k)} (\mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_{j-1})) \geq \mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_j)$.

Recall that $-A \leq x_j \leq A$. First, we assume $x_j \neq B$ so that $\mathrm{soc}(S' \times \tau) \times (\rho|\cdot|^{x_j})^k$ is irreducible. Since

$$[\mathrm{Jac}(\mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_{j-1}))] \geq \mathrm{soc}(S' \times \tau) \otimes (\rho|\cdot|^{x_j})^k \otimes \sigma_j,$$

there is $\tau' \in \mathrm{Irr}^{\mathrm{GL}}$ such that

$$(1) \quad [\mathrm{Jac}(\mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_{j-1}))] \geq \tau' \otimes \sigma_j$$

and

$$(2) \quad [\mathrm{Jac}(\tau')] \geq \mathrm{soc}(S' \times \tau) \otimes (\rho|\cdot|^{x_j})^k.$$

We show that these two inequalities combined imply that $\tau' = \mathrm{soc}(S' \times \tau) \times (\rho|\cdot|^{x_j})^k$. As in Paragraph 2.5, we use the multisegment notation: write $\tau' = L(\mathbf{m})$ and let $\lambda(\mathbf{m})$ be the associated standard module. Denote $\mathrm{soc}(S' \times \tau) = L(\Delta_\rho[a_1, b_1]^{k_1}, \dots, \Delta_\rho[a_r, b_r]^{k_r})$ with $b_1 < \cdots < b_{r-1} < b_r = -A - 1$ and $k_r = k$. By looking at the cuspidal support, as $x_j \geq b_u + 1$ for all $1 \leq u \leq r$, (2) implies that there exist $a_{u,v} \geq b_u$ for $1 \leq u \leq r$ and $1 \leq v \leq k_u$ such that

$$\sum_{u=1}^r \sum_{v=1}^{k_u} [a_{u,v}, b_u]_\rho \leq \mathbf{m}.$$

Suppose that $a_{u,v} \neq a_u$ for some (u, v) . Take the minimum u with this condition. Hence $a_{i,v} = a_i$ for any $i < u$ and $1 \leq v \leq k_i$. If $a_{u,v} > a_u$, then by Geometric Lemma (Theorem 2.1), we have $[\mathrm{Jac}(\lambda(\mathbf{m}))] \not\geq \mathrm{soc}(S' \times \tau) \otimes (\rho|\cdot|^{x_j})^k$. (See e.g. [29] for $\mathrm{Jac}(\mathrm{soc}(S' \times \tau))$.) This contradicts (2).

If $a_{u,v} < a_u$, we use (1). We get

$$\left(D_{\rho|\cdot|^{b_u}} \circ \cdots \circ D_{\rho|\cdot|^{a_{u,v}}} \right) \circ \left(\bigcirc_{1 \leq i < u} D_{\rho|\cdot|^{b_i}}^{(k_i)} \circ \cdots \circ D_{\rho|\cdot|^{a_i}}^{(k_i)} \right) (\mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_{j-1})) \neq 0.$$

Since $a_u < 0$ and $D_{\rho|\cdot|b_i} \circ \cdots \circ D_{\rho|\cdot|a_i}(\sigma_{j-1}) = 0$ for $i < u$, it implies that $D_{\rho|\cdot|b_u} \circ \cdots \circ D_{\rho|\cdot|a_{u,v}}(\sigma_{j-1}) \neq 0$. However, since $\sigma_{j-1} \leq [L_{j-1} \rtimes \sigma]$ by Lemma 7.6, we can find $b \geq b_u$ such that $D_{\rho|\cdot|b, \dots, \rho|\cdot|b_u}(\sigma) \neq 0$. This contradicts Condition 7.4 (1). Therefore, $a_{u,v} = a_u$ for any (u, v) , and hence

$$\sum_{u=1}^r k_u [a_u, b_u]_\rho \leq \mathbf{m}.$$

Comparing the cuspidal supports, we see that the difference is $k[x_j, x_j]_\rho$. Hence

$$\tau' = \text{soc}(S' \times \tau) \times (\rho|\cdot|^{x_j})^k \cong (\rho|\cdot|^{x_j})^k \times \text{soc}(S' \times \tau).$$

In particular, we have

$$[\text{Jac}(\text{soc}(\text{soc}(S' \times \tau) \rtimes \sigma_{j-1}))] \geq (\rho|\cdot|^{x_j})^k \otimes \text{soc}(S' \times \tau) \otimes \sigma_j.$$

It implies that there is $\pi'_j \in \text{Irr}^G$ such that

$$(3) \quad D_{\rho|\cdot|^{x_j}}^{(k)}(\text{soc}(\text{soc}(S' \times \tau) \rtimes \sigma_{j-1})) \geq \pi'_j$$

and

$$(4) \quad [\text{Jac}(\pi'_j)] \geq \text{soc}(S' \times \tau) \otimes \sigma_j.$$

We are going to show that $\pi'_j = \text{soc}(\text{soc}(S' \times \tau) \rtimes \sigma_j)$, which, together with (3), implies the assertion of the lemma. The idea of the proof is similar to the previous one, but we use (3) and (4) instead of (1) and (2). We write $\pi'_j = L(\mathbf{m}; \pi(\phi, \varepsilon))$ with $\pi(\phi, \varepsilon)$ tempered. Since $a_u < 0$, (4) implies that there exist $b_u \leq a_{u,v} \leq a_u$ such that

$$\sum_{u=1}^r \sum_{v=1}^{k_u} [a_{u,v}, b_u]_\rho \leq \mathbf{m}.$$

Suppose that $a_{u,v} < a_u$ for some (u, v) , and we take the minimum such u . Then by (3),

$$\left(D_{\rho|\cdot|b_u} \circ \cdots \circ D_{\rho|\cdot|a_{u,v}} \right) \circ \left(\bigcirc_{1 \leq i < u} D_{\rho|\cdot|b_i}^{(k_i)} \circ \cdots \circ D_{\rho|\cdot|a_i}^{(k_i)} \right) \circ D_{\rho|\cdot|^{x_j}}^{(k)}(\text{soc}(\text{soc}(S' \times \tau) \rtimes \sigma_{j-1})) \neq 0.$$

Since $\text{soc}(\text{soc}(S' \times \tau) \rtimes \sigma_{j-1}) \hookrightarrow \text{soc}(S' \times \tau) \rtimes \sigma_{j-1}$, we deduce that

$$\left(D_{\rho|\cdot|b_u} \circ \cdots \circ D_{\rho|\cdot|a_{u,v}} \right) \circ \left(\bigcirc_{1 \leq i < u} D_{\rho|\cdot|b_i}^{(k_i)} \circ \cdots \circ D_{\rho|\cdot|a_i}^{(k_i)} \right) \circ D_{\rho|\cdot|^{x_j}}^{(k)}(\text{soc}(S' \times \tau) \rtimes \sigma_{j-1}) \neq 0.$$

As $t_i > A + B$ for any $1 \leq i \leq k$, we have $D_{\rho|\cdot|^{\pm x_j}}(\text{soc}(S' \times \tau)) = 0$. Hence we get:

$$\left(D_{\rho|\cdot|b_u} \circ \cdots \circ D_{\rho|\cdot|a_{u,v}} \right) \circ \left(\bigcirc_{1 \leq i < u} D_{\rho|\cdot|b_i}^{(k_i)} \circ \cdots \circ D_{\rho|\cdot|a_i}^{(k_i)} \right) (\text{soc}(S' \times \tau) \rtimes D_{\rho|\cdot|^{x_j}}^{(k)}(\sigma_{j-1})) \neq 0.$$

By the second bullet in Lemma 7.6 and Tadić's formula (Corollary 2.2), we deduce that:

$$D_{\rho|\cdot|b_u} \circ \cdots \circ D_{\rho|\cdot|a_{u,v}} \left(\left(\bigcirc_{1 \leq i < u} D_{\rho|\cdot|b_i}^{(k_i)} \circ \cdots \circ D_{\rho|\cdot|a_i}^{(k_i)} (\text{soc}(S' \times \tau)) \right) \rtimes D_{\rho|\cdot|^{x_j}}^{(k)}(\sigma_{j-1}) \right) \neq 0.$$

However, since

$$D_{\rho|\cdot|^{\pm x}} \left(\bigcirc_{1 \leq i < u} D_{\rho|\cdot|b_i}^{(k_i)} \circ \cdots \circ D_{\rho|\cdot|a_i}^{(k_i)} (\text{soc}(S' \times \tau)) \right) = 0$$

for all $b_u \leq x \leq a_{u,v}$, by Tadić's formula (Corollary 2.2), we get

$$D_{\rho|\cdot|^{b_u}} \circ \cdots \circ D_{\rho|\cdot|^{a_{u,v}}} \circ D_{\rho|\cdot|^{x_j}}^{(k)}(\sigma_{j-1}) \neq 0.$$

This contradicts the second bullet in Lemma 7.6. Therefore, $a_{u,v} = a_u$ for any (u, v) , and hence

$$\sum_{u=1}^r k_u[a_u, b_u]_\rho \leq \mathbf{m}.$$

Let $\mathbf{m}' = \mathbf{m} - \sum_{u=1}^r k_u[a_u, b_u]_\rho$. Let us similarly prove that $L(\mathbf{m}) = \mathrm{soc}(\mathrm{soc}(S' \times \tau) \times L(\mathbf{m}'))$. Write $\mathbf{m}' = [a'_1, b'_1]_\rho + \cdots + [a'_{r'}, b'_{r'}]_\rho$, with $b'_1 < \cdots < b'_{r'}$. Assume for the sake of contradiction that $[a'_1, b'_1]_\rho$ precedes one of the segments $[a_i, b_i]_\rho$'s and let u be the smallest i such that $[a'_1, b'_1]_\rho$ precedes $[a_i, b_i]_\rho$. Then by (3), we deduce that

$$\left(D_{\rho|\cdot|^{b'_1}} \circ \cdots \circ D_{\rho|\cdot|^{a'_1}} \right) \circ \left(\bigcirc_{1 \leq i < u} D_{\rho|\cdot|^{b_i}}^{(k_i)} \circ \cdots \circ D_{\rho|\cdot|^{a_i}}^{(k_i)} \right) \circ D_{\rho|\cdot|^{x_j}}^{(k)}(\mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_{j-1})) \neq 0.$$

This, just as before, contradicts the second bullet in Lemma 7.6 and proves our claim. Hence we can write $\pi'_j = \mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma'_j)$ for some $\sigma'_j \in \mathrm{Irr}^G$. Since

$$\pi'_j \leq D_{\rho|\cdot|^{x_j}}^{(k)}(\mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_{j-1})),$$

we see that σ'_j is uniquely determined by the inequality

$$[\mathrm{Jac}(\pi'_j)] \geq \Delta_\rho[a_1, b_1]^{k_1} \otimes \cdots \otimes \Delta_\rho[a_r, b_r]^{k_r} \otimes \sigma'_j.$$

Hence we have $\sigma'_j = \sigma_j$, and we conclude that $\pi'_j = \mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_j)$. Therefore

$$D_{\rho|\cdot|^{x_j}}^{(k)}(\mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_{j-1})) \geq \mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_j),$$

as desired. We obtain the claim for j with $x_j \neq B$.

Next, we assume that $x_j = B$. By the proof of [6, Proposition 3.4], we have

$$\mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_{j-1}) = \mathrm{soc}(S' \rtimes \mathrm{soc}(\tau \rtimes \sigma_{j-1})) = \mathrm{soc}(S' \rtimes \pi_{j-1}).$$

Hence by the same argument starting with

$$[\mathrm{Jac}(\mathrm{soc}(S' \rtimes \pi_{j-1}))] \geq S' \otimes (\rho|\cdot|^{x_j})^k \otimes \pi_j,$$

we have

$$D_{\rho|\cdot|^{x_j}}^{(k)}(\mathrm{soc}(S' \rtimes \pi_{j-1})) \geq \mathrm{soc}(S' \rtimes \pi_j).$$

Since $\mathrm{soc}(S' \rtimes \pi_j) = \mathrm{soc}(S' \rtimes \mathrm{soc}(\tau \rtimes \sigma_j)) = \mathrm{soc}(\mathrm{soc}(S' \times \tau) \rtimes \sigma_j)$, we obtain the claim for j with $x_j = B$. This completes the proof of Lemma 7.5. $\square$

7.3. Connection to Mœglin's construction. In this subsection, we continue under the assumptions of the previous subsection. Namely, we assume that σ satisfies Condition 7.4. Moreover, we also assume that σ is of Arthur type. We will show that these two assumptions alone impose strong constraints on the Langlands data of σ (Lemma 7.7) and, in many cases, on the extended multisegment attached to it (Proposition 7.8).

In this subsection, we assume that σ satisfies Condition 7.4 (2). By Aubert duality, we can obtain an analogous result when Condition 7.4 (1) is assumed.

Under Condition 7.4 (2), we have

$$D_{\rho|\cdot|^{b, \dots, \rho|\cdot|^{a}}}(\sigma) = 0$$

for any $b \leq a$ such that $a > A+1$, or $b \geq B+1$ and $a = A+1$. By [7, Theorem 4.1], there exists an extended multisegment $\mathcal{E}$ with $\sigma = \pi(\mathcal{E})$ such that for $([A', B']_\rho, *, *)$, $([A'', B'']_\rho, *, *) \in \mathcal{E}$, if $[A', B']_\rho \subset [A, B]_\rho \not\supset [A'', B'']_\rho$, then $([A', B']_\rho, *, *) > ([A'', B'']_\rho, *, *)$.

We use Lemma 7.5 (2) for suitable integers $t_1, \dots, t_k \gg 0$. We will give conditions for these integers later. Define S' as in this lemma. By [6, Theorem 1.1], in the notation of Lemma 7.5 (2), any irreducible summand π'' in $\text{soc}(\text{soc}(S' \times \tau \times S) \rtimes \sigma)$ is of Arthur type, and has an extended multisegment of the form

$$\mathcal{E}'' = \mathcal{E} \cup \bigcup_{i=1}^k \{([A, B]_\rho, l, \eta_i), ([A + t_i, B + t_i]_\rho, l, \eta'_i)\},$$

where the order $<$ on $\mathcal{E}''$ is given by

$$\begin{aligned} ([A', B']_{\rho'}, l', \eta') &< ([A, B]_\rho, l, \eta_1) < \dots < ([A, B]_\rho, l, \eta_k) \\ &< ([A + t_1, B + t_1]_\rho, l, \eta'_1) < \dots < ([A + t_k, B + t_k]_\rho, l, \eta'_k) \end{aligned}$$

for any $([A', B']_{\rho'}, l', \eta') \in \mathcal{E}$, and the restriction of $<$ on $\mathcal{E}$ is the same as the original order on it. Notice that this is an admissible order since if $\rho' \cong \rho$ and $A' = A+1$, then $[A', B']_{\rho'} \supset [A, B]_\rho$. By Lemma 7.5 (2), we can choose such an $\mathcal{E}''$ such that

$$D_S(\pi(\mathcal{E}'')) \geq \text{soc}(S' \rtimes \pi).$$

We write

$$\mathcal{E} \cup \{([A, B]_\rho, l, \eta_i) \mid i = 1, \dots, k\} = \{([A'_j, B'_j]_{\rho'}, l'_j, \eta'_j) \mid j = 1, \dots, k'\},$$

which is independent of $t_1, \dots, t_k$. Now we choose integers $t_1, \dots, t_k > A - B$ so that there are integers $t'_1, \dots, t'_{k'} \geq 0$ such that

$$\mathcal{E}''_{\gg} = \{([A'_j + t'_j, B'_j + t'_j]_{\rho'}, l'_j, \eta'_j) \mid j = 1, \dots, k'\} \cup \{([A + t_i, B + t_i]_\rho, l, \eta'_i) \mid i = 1, \dots, k\}$$

has a non-negative DDR, i.e.,

$$\begin{aligned} 0 \leq B'_1 + t'_1 &\leq A'_1 + t'_1 < \dots < B'_{k'} + t'_{k'} \leq A'_{k'} + t'_{k'} \\ &< B + t_1 \leq A + t_1 < \dots < B + t_k \leq A + t_k. \end{aligned}$$

In particular, $B + t_1 > A$. Then by construction together with [10, Theorem 7.1], one can write

$$\pi(\mathcal{E}'') = L(\mathbf{m}_1 + \mathbf{m}_0; \pi(\phi_1 \oplus \phi_0, \varepsilon)),$$

where

- $\mathbf{m}_1$ is the multisegment given by

$$\mathbf{m}_1 = \sum_{i=1}^k \sum_{j=0}^{l-1} [B + t_i + j, -A - t_i + j]_\rho;$$

- ϕ_1 is the representation of $W_F \times \text{SL}_2(\mathbb{C})$ given by

$$\phi_1 = \bigoplus_{i=1}^k \bigoplus_{j=0}^{A-B-2l} \rho \boxtimes S_{2(B+t_i+l+j)+1};$$

- ε satisfies that

$$\varepsilon(\rho \boxtimes S_{2(B+t_i+l+j)+1}) = (-1)^j \eta'_i$$

for $1 \leq i \leq k$ and $0 \leq j \leq A - B - 2l$;

- $\mathbf{m}_0$ is a multisegment such that if $[x, -y]_\rho \in \mathbf{m}_0$, then $x < y < B + t_1$;
- ϕ_0 is a tempered representation of $W_F \times \mathrm{SL}_2(\mathbb{C})$ such that if $\rho \boxtimes S_{2d+1} \subset \phi_0$, then $d < B + t_1$.

Lemma 7.7. *We have $\mathrm{soc}(S' \rtimes \pi) = L(\mathbf{m}_1 + \mathbf{m}'_0; \pi(\phi_1 \oplus \phi'_0, \varepsilon'))$ for some $\mathbf{m}'_0, \phi'_0, \varepsilon'$ with $\varepsilon'|_{A_{\phi_1}} = \varepsilon|_{A_{\phi_1}}$.*

Proof. Recall that $\mathcal{E}''$ is chosen so that $D_S(\pi(\mathcal{E}'')) \geq \mathrm{soc}(S' \rtimes \pi)$. Hence we can find $\tau \in \mathrm{Irr}^{\mathrm{GL}}$ such that $[\mathrm{Jac}(\pi(\mathcal{E}''))] \geq \tau \otimes \mathrm{soc}(S' \rtimes \pi)$ and $[\mathrm{Jac}(\tau)] \geq (\rho| \cdot |^{x_1})^k \otimes \cdots \otimes (\rho| \cdot |^{x_{ab}})^k$. Since $t_i \gg 0$, we see that $\pi(\mathcal{E}'') = \mathrm{soc}(L(\mathbf{m}_1) \rtimes L(\mathbf{m}_0; \pi(\phi_1 \oplus \phi_0, \varepsilon)))$. It is the image of any nonzero equivariant homomorphism

$$L(\mathbf{m}_1)^\vee \rtimes L(\mathbf{m}_0; \pi(\phi_1 \oplus \phi_0, \varepsilon)) \rightarrow L(\mathbf{m}_1) \rtimes L(\mathbf{m}_0; \pi(\phi_1 \oplus \phi_0, \varepsilon)).$$

Since $\mathrm{supp}(L(\mathbf{m}_1) \rtimes L(\mathbf{m}_1)^\vee) \cap \mathrm{supp}(S) = \emptyset$, by the functoriality of Geometric Lemma (Theorem 2.1), $\tau \otimes \mathrm{soc}(S' \rtimes \pi)$ should appear in the image of the induced map

$$(\mathbf{1}_{\mathrm{GL}_0} \otimes L(\mathbf{m}_1)^\vee) \rtimes \mathrm{Jac}(L(\mathbf{m}_0; \pi(\phi_1 \oplus \phi_0, \varepsilon))) \rightarrow (\mathbf{1}_{\mathrm{GL}_0} \otimes L(\mathbf{m}_1)) \rtimes \mathrm{Jac}(L(\mathbf{m}_0; \pi(\phi_1 \oplus \phi_0, \varepsilon)))$$

as a subquotient. This means that $\mathrm{soc}(S' \rtimes \pi) = \mathrm{soc}(L(\mathbf{m}_1) \rtimes \pi_0)$ for some $\pi_0 \in \mathrm{Irr}^G$ such that

$$[\mathrm{Jac}(L(\mathbf{m}_0; \pi(\phi_1 \oplus \phi_0, \varepsilon)))] \geq \tau \otimes \pi_0.$$

Hence $\pi_0 \leq D_S(L(\mathbf{m}_0; \pi(\phi_1 \oplus \phi_0, \varepsilon)))$. Since $t_1, \dots, t_k \gg 0$, we see that $\pi_0 = L(\mathbf{m}'_0; \pi(\phi_1 \oplus \phi'_0, \varepsilon'))$ for some $\mathbf{m}'_0$ and (ϕ'_0, ε') . Therefore $\mathrm{soc}(S' \rtimes \pi) = L(\mathbf{m}_1 + \mathbf{m}'_0; \pi(\phi_1 \oplus \phi'_0, \varepsilon'))$, as desired. $\square$

Recall that σ is of Arthur type.

Proposition 7.8. *Continue in the above setting. Suppose furthermore that $D_{\rho| \cdot |^b, \dots, \rho| \cdot |^a}(\sigma) = 0$ for any $a \geq b > 0$ such that $a > A$. There is an extended multisegment $\mathcal{E}_0$ such that*

- $\pi(\mathcal{E}_0) = \sigma$;
- $\mathcal{E}_0$ contains $\{([A, B]_\rho, l, \eta'_i) \mid i = 1, \dots, k\}$ as multisets;
- if $([A', B']_\rho, l', \eta') \in \mathcal{E}_0$ is not in $\{([A, B]_\rho, l, \eta'_i) \mid i = 1, \dots, k\}$, then $([A', B']_\rho, l', \eta') < ([A, B]_\rho, l, \eta'_i)$.

Proof. By the additional assumption, there is an extended multisegment $\mathcal{E}$ with $\sigma = \pi(\mathcal{E})$ such that for $([A', B']_\rho, *, *)$, $([A'', B'']_\rho, *, *) \in \mathcal{E}$, if $A' \leq A$, $B' > 0$ and $A'' > A$, then $([A', B']_\rho, *, *) > ([A'', B'']_\rho, *, *)$. We choose such an $\mathcal{E}$ so that the number

$$X = \sum_{\substack{([A', B']_\rho, l', \eta') \in \mathcal{E} \\ A' \leq A, B' > 0}} A' - B' + 1$$

is maximal. By changing the admissible order if necessary, we may assume that we can write

$$\mathcal{E} = \mathcal{E}_1 \cup \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i=1, \dots, m},$$

where

- $\{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i=1, \dots, m}$ is the multiset contained in $\mathcal{E}$ consisting of $([A', B']_\rho, *, *)$ such that $A' \leq A$ and $B' > 0$;
- $A_1 \leq \dots \leq A_m$;
- $\mathcal{E}_1$ is the multiset contained in $\mathcal{E}$ consisting of other $([A', B']_\rho, *, *)$;

- the admissible order is so that

$$([A', B']_{\rho'}, *, *) < ([A_i, B_i]_{\rho}, l_i, \eta_i)$$

for $([A', B']_{\rho'}, *, *) \in \mathcal{E}_1$, and the canonical order on $\{1, \dots, m\}$.

Here, m can be equal to zero.

For Lemma 7.7 for $t_1, \dots, t_k \gg 0$ to be valid, namely for $\mathfrak{m}_1$ and $(\phi_1, \varepsilon|_{A_{\phi_1}})$ appearing in the Langlands data of $\text{soc}(S' \rtimes \pi) = \text{soc}(\text{soc}(S' \times \tau) \rtimes \sigma)$, it is necessary that by [5, Theorem 5.2], one can change $\{([A_i, B_i]_{\rho}, l_i, \eta_i)\}_{i=1, \dots, m}$ so that it contains k extended segments of the form $([A, B]_{\rho}, *, *)$. This is the $\mathcal{E}_0$ we need to find. $\square$

7.4. Proof of Theorem 4.7. In this paragraph, we finally prove Theorem 4.7. Recall our setting. Let π be an irreducible representation of G which has SPS, and let

$$(\tau_{\text{bad}}^-, \{\tau_j^-\}_{1 \leq j \leq s}, \{\tau_j^+\}_{1 \leq j \leq r}, \{\pi_j\}_{0 \leq j \leq r+s})$$

be the SZ-decomposition of π so that

$$\pi \hookrightarrow \tau_{\text{bad}}^- \times \tau_s^- \times \dots \times \tau_1^- \times \tau_r^+ \times \dots \times \tau_1^+ \rtimes \pi_0.$$

Fix $\rho \in \mathcal{C}_{\text{sd}}^{\text{GL}}$ and $A > -1$. Write

$$\begin{aligned} \{j \in \{1, \dots, r\} \mid \tau_j^+ = Z_{\rho}[B_j + 1, A + 1]^{k_j} \text{ for some } -1 < B_j \leq A \text{ and } k_j > 0\} \\ = \{i, i + 1, \dots, i'\}. \end{aligned}$$

As an inductive hypothesis, suppose that π_{i-1} is of Arthur type. Since $\tau_i^+ \times \dots \times \tau_{i+1}^+ \times \tau_i^+$ is irreducible, we have

$$\pi_{i'} = \text{soc}(\tau_{i'}^+ \times \dots \times \tau_{i+1}^+ \times \tau_i^+ \rtimes \pi_{i-1}).$$

We apply Proposition 7.3 to $S = S_{i'} \times \dots \times S_{i+1} \times S_i$, where

$$S_j = \begin{pmatrix} B_j & \dots & A \\ \vdots & \ddots & \vdots \\ -A & \dots & -B_j \end{pmatrix}_{\rho}^{k_j}.$$

Notice that $\sigma = \pi_{i-1}$ and $\text{soc}(\tau_j^+ \rtimes \pi_{i-1})$ satisfy Condition 7.4 (2) for $i \leq j \leq i'$. We choose $t_{j,1}, \dots, t_{j,k_j} \gg 0$ for $i \leq j \leq i'$ such that

$$\begin{aligned} 0 \ll B_{i'} + t_{i',1} \leq A + t_{i',1} < \dots < B_{i'} + t_{i',k_{i'}} \leq A + t_{i',k_{i'}} \\ < \dots < B_i + t_{i,1} \leq A + t_{i,1} < \dots < B_i + t_{i,k_i} \leq A + t_{i,k_i} \end{aligned}$$

and set

$$S'_j = \begin{pmatrix} B_j + t_{j,1} & \dots & A + t_{j,1} \\ \vdots & \ddots & \vdots \\ B_j + 2 & \dots & A + 2 \end{pmatrix}_{\rho} \times \dots \times \begin{pmatrix} B_j + t_{j,k_j} & \dots & A + t_{j,k_j} \\ \vdots & \ddots & \vdots \\ B_j + 2 & \dots & A + 2 \end{pmatrix}_{\rho}.$$

We repeatedly apply Lemma 7.7 to

$$\text{soc}(S'_{i'} \times \dots \times S'_{i+1} \times S'_i \rtimes \pi_{i'})$$

with respect to S'_j for $j = i, i + 1, \dots, i'$, and we obtain some information about the Langlands data for this representation. As a conclusion, applying the argument of Proposition 7.8

successively for $j = i', i' - 1, \dots, i$, in its relative form that preserves the rows already obtained, we can find an extended multisegment $\mathcal{E}_{i-1}$ with $\pi(\mathcal{E}_{i-1}) = \pi_{i-1}$ such that

- $\mathcal{E}_{i-1}$ contains k_j extended segments of the form $([A, B_j]_\rho, *, *)$ for $i \leq j \leq i'$;
- if $([A', B']_\rho, *, *) \in \mathcal{E}_{i-1}$ is another extended segment, then

$$([A', B']_\rho, *, *) < ([A, B_{i'}]_\rho, *, *) < \dots < ([A, B_{i+1}]_\rho, *, *) < ([A, B_i]_\rho, *, *).$$

Indeed, suppose that the required rows $([A, B_{j'}]_\rho, *, *)$ with multiplicity $k_{j'}$ have already been obtained for $j < j' \leq i'$. In the proof of Proposition 7.8 for $B = B_j$, each application of [5, Theorem 5.2] involves two rows whose underlying segments are of the form $[A, C]_\rho$ and $[D, B_j]_\rho$ with $C < B_j$ and $D > A$, and replaces them by $[D, C]_\rho$ and $[A, B_j]_\rho$. Since $B_{j'} > B_j$ for $j' > j$, none of the previously obtained rows $[A, B_{j'}]_\rho$ is involved in this operation. Repeating this argument k_j times produces the required k_j rows with underlying segment $[A, B_j]_\rho$. By the descending induction on j , we obtain $\mathcal{E}_{i-1}$.

Then for $i \leq i_1 \leq i'$, we have $\pi_{i_1} = \pi(\mathcal{E}_{i_1})$, where $\mathcal{E}_{i_1}$ is defined from $\mathcal{E}_{i-1}$ by replacing k_j extended segments $([A, B_j]_\rho, *, *)$ with $([A + 1, B_j + 1]_\rho, *, *)$ for $i \leq j \leq i_1$. In particular, by induction, we see that π_r is of Arthur type. After taking Aubert duality, the same argument works for π_j for $j > r$. We conclude that π_{r+s} is of Arthur type. This completes the proof of Theorem 4.7.

Now, all claims in Section 4 are established. Let us look back at our proof for the following example.

Example 7.9. *We use the same notation as in Example 7.1. Let*

$$\sigma = \pi \begin{pmatrix} -1 & 0 & 1 & 2 \\ \triangleleft & \oplus & \ominus & \triangleright \\ & & \ominus & \end{pmatrix} = L(\Delta[-1, -2]; \pi(0^+, 1^-, 1^-)) \in \mathrm{Irr}(\mathrm{Sp}_{10}(F)).$$

We show that

$$\pi = \mathrm{soc}(| \cdot |^{-2} \rtimes \sigma) = L(| \cdot |^{-2}, \Delta[-1, -2]; \pi(0^+, 1^-, 1^-)) \in \mathrm{Irr}(\mathrm{Sp}_{12}(F))$$

is not unitary, or even does not have SPS.

First of all, the SZ decomposition of π is given by

$$\pi \hookrightarrow | \cdot |^{-2} \times \Delta[-1, -2] \times | \cdot |^1 \rtimes L(\Delta[0, -1], \pi(0^+)).$$

With $\pi_0 = L(\Delta[0, -1], \pi(0^+))$, we have

$$\pi_0 = \pi \begin{pmatrix} 0 & 1 \\ \oplus & \ominus \\ \ominus & \end{pmatrix}, \quad \pi_1 = \mathrm{soc}(| \cdot |^1 \rtimes \pi_0) = \pi \begin{pmatrix} 0 & 1 \\ \oplus & \ominus \\ \ominus & \oplus \end{pmatrix}.$$

By [5, Theorem 6.2], the Aubert dual of π_1 is

$$\hat{\pi}_1 = \hat{\pi} \begin{pmatrix} 0 & 1 \\ \oplus & \ominus \\ \ominus & \end{pmatrix} = \pi \begin{pmatrix} -1 & 0 & 1 \\ \triangleleft & \ominus & \triangleright \\ & \ominus & \oplus \end{pmatrix}.$$

If we denote this representation by $\hat{\pi}_1$, then we have $\hat{\pi} = \text{soc}(|\cdot|^2 \times Z[1, 2] \rtimes \hat{\pi}_1)$. For $t_2, t_3 \gg 0$ with $1 + t_2 > 2 + t_3 > 2$, we have

$$\begin{aligned} & \text{soc} \left(\begin{pmatrix} 2+t_3 \\ \vdots \\ 3 \end{pmatrix} \times \begin{pmatrix} 1+t_2 & 2+t_2 \\ \vdots & \vdots \\ 2 & 3 \end{pmatrix} \rtimes \hat{\pi} \right) \\ &= \text{soc} \left(\begin{pmatrix} 2+t_3 \\ \vdots \\ 2 \end{pmatrix} \rtimes \pi \begin{pmatrix} -1 & 0 & 1 & \dots & 1+t_2 & 2+t_2 \\ \triangleleft & \ominus & \triangleright & & \ominus & \oplus \end{pmatrix} \right) \\ &= L(\Delta[-1, -(2+t_3)], \pi(0^-, (1+t_2)^-, (2+t_2)^+)). \end{aligned}$$

This does not satisfy Lemma 7.7. Hence $\hat{\pi}$ and hence π do not have SPS.

Note that

$$\hat{\pi}_2 = \text{soc}(Z[1, 2] \rtimes \hat{\pi}_1) = \pi \begin{pmatrix} -1 & 0 & 1 & 2 \\ \triangleleft & \ominus & \triangleright & \\ & & \ominus & \oplus \end{pmatrix}.$$

For $t_3 > 0$, we have

$$\text{soc}(\Delta[2+t_3, 3] \rtimes \hat{\pi}) = \text{soc}(\Delta[2+t_3, 2] \rtimes \hat{\pi}_2) = L(\Delta[1, -2]; \pi(0^-, 1^-, (2+t_3)^+)),$$

which satisfies Lemma 7.7.

8. A CASE BEYOND GOOD PARITY

In this section, we study a case beyond the good parity setting. Although limited, this extension may be useful for applications, especially global ones.

8.1. Statement. We know that if $\psi \in \Psi(G)$, then all members in Π_ψ are unitary. However, due to the lack of a proof of the generalized Ramanujan conjecture, if Ψ is a global A -parameter of a classical group $\mathbb{G}$ over a number field $\mathbb{F}$, and if v is a place of $\mathbb{F}$ such that $\mathbb{F}_v = F$ and $G = \mathbb{G}(\mathbb{F}_v)$, then the localization $\psi = \Psi_v$ may not lie in $\Psi(G)$. For the global A -parameters and their localization, see [3, Section 1.4, Theorem 1.4.2].

By the weak Ramanujan bound, which is known, $\psi = \Psi_v$ takes the form

$$\psi = \psi_0 \oplus \bigoplus_{i=1}^r (\rho_i |\cdot|^{x_i} \oplus \rho_i^\vee |\cdot|^{-x_i}) \boxtimes S_{c_i} \boxtimes S_{d_i},$$

where

- $\psi_0 = \psi_{\text{good}}$ is an A -parameter of good parity;
- $\rho_i \in \mathcal{C}_{\text{unit}}^{\text{GL}}$;
- $0 \leq x_i < \frac{1}{2}$;
- if $x_i = 0$, then $\rho_i \boxtimes S_{c_i} \boxtimes S_{d_i}$ is not self-dual of the same type as ψ .

(Here, r may be zero, in which case $\psi = \psi_{\text{good}}$ is an A -parameter of good parity.) For such a parameter ψ , its A -packet Π_ψ is defined as the set of irreducible parabolically induced representations

$$\pi = \bigtimes_{i=1}^r \text{Sp}(\rho_i, c_i, d_i) |\cdot|^{x_i} \rtimes \pi_0$$

for $\pi_0 \in \Pi_{\psi_0}$.

In this section, we prove the following theorem. It was also proven in [25, Theorem 1.3] by the same argument almost simultaneously.

Theorem 8.1. *Let*

$$\pi = \bigtimes_{i=1}^r \mathrm{Sp}(\rho_i, c_i, d_i) \cdot |^{x_i} \rtimes \pi_0$$

where $\pi_0 \in \mathrm{Irr}_{\mathrm{good}}^G$ is of Arthur type, $\rho_i \in \mathcal{C}_{\mathrm{unit}}^{\mathrm{GL}}$, and $0 \leq x_i < \frac{1}{2}$ for $1 \leq i \leq r$. Suppose that π is irreducible. For each $\rho \in \mathcal{C}_{\mathrm{unit}}^{\mathrm{GL}}$ and each pair of positive integers (c, d) , we set

$$I(\rho, c, d) = \{i \in \{1, \dots, r\} \mid \rho_i \cong \rho, (c_i, d_i) = (c, d)\}.$$

Then π is unitary if and only if for each ρ and (c, d) , the following conditions hold:

- If $\rho \not\cong \rho^\vee$, then

$$\{x_i \mid i \in I(\rho, c, d), x_i \neq 0\} = \{x_i \mid i \in I(\rho^\vee, c, d), x_i \neq 0\}$$

as multisets.

- If $\rho \cong \rho^\vee$ and $|I(\rho, c, d)|$ is odd, then the unitary induced representation $\mathrm{Sp}(\rho, c, d) \rtimes \pi_0$ is irreducible.

Remark 8.2. Let π be as in Theorem 8.1. The Langlands data for π can be computed from the ones of π_0 by [47, Proposition 1.3] and the irreducibility of the parabolic induction. Conversely, the expression $\pi = \times_{i=1}^r \mathrm{Sp}(\rho_i, c_i, d_i) \cdot |^{x_i} \rtimes \pi_0$ is determined uniquely (up to isomorphism) by the Langlands data of π . In particular, the set $I(\rho, c, d)$ is well-defined.

The proof of Theorem 8.1 is divided into the next two subsections. In Section 8.2 (resp. Section 8.3), we prove the “if” direction (resp. “only if” direction). Throughout, we use (UI), (UR), (CS), and (RP1) from Proposition 2.4 repeatedly.

8.2. Proof of unitarity. Assume that π satisfies the conditions in Theorem 8.1. We will show that π is unitary. The proof is by induction on r .

First, suppose that $x_i = 0$ for some i . Since π is irreducible, by [19, Theorem 1.2], we see that $\mathrm{Sp}(\rho_i, c_i, d_i) \rtimes \pi_0$ is also irreducible. Then since $\pi' = \times_{j \neq i} \mathrm{Sp}(\rho_j, c_j, d_j) \cdot |^{x_j} \rtimes \pi_0$ is unitary by the induction hypothesis, so is $\pi = \mathrm{Sp}(\rho_i, c_i, d_i) \rtimes \pi'$ by (UI). In the rest of this subsection, we assume that $x_i > 0$ for all i .

Fix $\rho \in \mathcal{C}_{\mathrm{unit}}^{\mathrm{GL}}$ and a pair (c, d) . Suppose that $r \geq 2$ and that there exist i and j with $i \neq j$ such that $\rho_i \cong \rho$, $\rho_j \cong \rho^\vee$ and $(c_i, d_i) = (c_j, d_j) = (c, d)$. We may assume that $(i, j) = (1, 2)$, $x_1 \geq x_2$ and that $x_1 = x_2$ if $\rho \not\cong \rho^\vee$. By induction hypothesis,

$$\pi' = \bigtimes_{i=3}^r \mathrm{Sp}(\rho_i, c_i, d_i) \cdot |^{x_i} \rtimes \pi_0$$

is unitary. Since $0 < x_2 < \frac{1}{2}$, by [43], $\mathrm{Sp}(\rho, c, d) \cdot |^{x_2} \times \mathrm{Sp}(\rho, c, d) \cdot |^{-x_2}$ is unitary. Hence

$$\mathrm{Sp}(\rho, c, d) \cdot |^{x_2} \times \mathrm{Sp}(\rho, c, d) \cdot |^{-x_2} \rtimes \pi' \cong \mathrm{Sp}(\rho, c, d) \cdot |^{x_2} \times \mathrm{Sp}(\rho^\vee, c, d) \cdot |^{x_2} \rtimes \pi'$$

is an irreducible unitary representation by (UI). In particular, if $\rho \not\cong \rho^\vee$, then π is unitary. If $\rho \cong \rho^\vee$ and if we set

$$\Pi_t = \mathrm{Sp}(\rho, c, d) \cdot |^t \times \mathrm{Sp}(\rho, c, d) \cdot |^{x_2} \rtimes \pi',$$

then for $x_2 \leq t \leq x_1$, we see that Π_t is an irreducible Hermitian representation since $\bar{\rho} \cong \rho^\vee \cong \rho$. Since Π_{x_2} is unitary, by (CS), we see that $\pi = \Pi_{x_1}$ is also unitary.

Hence we may assume that $\rho_i \cong \rho_i^\vee$ for all $1 \leq i \leq r$, and that $I(\rho, c, d)$ is a singleton or empty for each $\rho \in \mathcal{C}_{\text{sd}}^{\text{GL}}$ and (c, d) . For $(t_1, \dots, t_r)$ with $0 \leq t_i \leq x_i$, we consider

$$\Pi_{(t_1, \dots, t_r)} = \bigotimes_{i=1}^r \text{Sp}(\rho_i, c_i, d_i) \cdot |^{t_i} \rtimes \pi_0.$$

By assumption together with [19, Theorem 1.2], we see that $\Pi_{(0, \dots, 0)}$ is irreducible. In particular, $\Pi_{(t_1, \dots, t_r)}$ is an irreducible Hermitian representation for any $(t_1, \dots, t_r)$ with $0 \leq t_i \leq x_i$. Since $\Pi_{(0, \dots, 0)}$ is unitary, by (CS), we see that $\pi = \Pi_{(x_1, \dots, x_r)}$ is also unitary. This completes the proof of the “if” part of Theorem 8.1.

8.3. Proof of non-unitarity. In this subsection, we prove the “only if” part. In the proof, we will need to establish the non-vanishing of certain Arthur-type representations attached to extended multisegments. The criterion of Bošnjak–Stadler [19] is better adapted for this purpose, but their conventions for extended multisegments differ slightly from ours. For the convenience of the reader, we briefly recall them here. The following is taken from [19, Definitions 3.3, 3.4, 3.15].

Definition 8.3. (1) A *modified multisegment* for G is an object of the form

$$\mathcal{S} = \{([A_i, B_i]_{\rho_i}, \mu_i)\}_{i \in (I, >)},$$

where, for $\{([A_i, B_i]_{\rho_i})\}_{i \in (I, >)}$, the same conditions as in Definition 5.1 are imposed, and where $\mu_i \in \mathbb{Z}$ satisfies

$$|\mu_i| \leq b_i = A_i - B_i + 1, \quad \mu_i \equiv b_i \pmod{2},$$

and such that the sign condition

$$\sum_{i \in I} \left(\left\lceil \frac{\mu_i}{2} \right\rceil + \mu_i \sum_{\substack{j \in I, j < i \\ \rho_j \cong \rho_i}} (b_j - 1) \right) \equiv 0 \pmod{2}$$

holds.

(2) Define a map

$$\mathcal{F}: \mathcal{S} = \{([A_i, B_i]_{\rho_i}, \mu_i)\}_{i \in (I, >)} \mapsto \mathcal{E} = \{([A_i, B_i]_{\rho_i}, l_i, \eta_i)\}_{i \in (I, >)}$$

by

$$l_i = \frac{b_i - |\mu_i|}{2} \quad \text{and} \quad \eta_i = \delta_i \text{sgn}(\mu_i),$$

where

$$\delta_i = \prod_{\substack{j \in I, j < i \\ \rho_j \cong \rho_i}} (-1)^{b_j - 1},$$

and

$$\text{sgn}: \mathbb{Z} \rightarrow \{\pm 1\}, \quad \text{sgn}(\nu) = \begin{cases} 1 & \text{if } \nu \geq 0, \\ -1 & \text{if } \nu < 0. \end{cases}$$

Its inverse map is given by

$$\mu_i = \delta_i \eta_i (b_i - 2l_i).$$

(3) Set $\pi(\mathcal{S}) = \pi(\mathcal{F}(\mathcal{S}))$.

(4) We say that $\mathcal{S} = \{([A_i, B_i]_{\rho_i}, \mu_i)\}_{i \in (I, >)}$ is *standard* if, for $i, j \in I$ with $\rho_i \cong \rho_j$,

$$B_i < B_j, \text{ or } B_i = B_j \text{ and } A_i > A_j \implies i < j.$$

Suppose that $\pi = \times_{i=1}^r \mathrm{Sp}(\rho_i, c_i, d_i) \cdot |^{x_i} \rtimes \pi_0$ is unitary. Our goal is to show that the conditions in Theorem 8.1 hold. The proof proceeds by induction on r .

First, assume that $x_i = 0$ for some i . Put $\pi' = \times_{j \neq i} \mathrm{Sp}(\rho_j, c_j, d_j) \cdot |^{x_j} \rtimes \pi_0$. Then $\mathrm{Sp}(\rho_i, c_i, d_i) \boxtimes \pi'$ is an irreducible Hermitian representation of a Levi subgroup of G such that its induction $\mathrm{Sp}(\rho_i, c_i, d_i) \rtimes \pi'$ is irreducible and unitary. Hence by (UR), we see that $\mathrm{Sp}(\rho_i, c_i, d_i) \boxtimes \pi'$ is also unitary. In particular, π' is unitary. Repeating this argument, we may assume that $x_i > 0$ for $1 \leq i \leq r$.

Next, assume that $\rho \not\cong \rho^\vee$. Since π is Hermitian, we see that

$$\begin{aligned} \bar{\pi} &\cong \times_{i=1}^r \mathrm{Sp}(\bar{\rho}_i, c_i, d_i) \cdot |^{x_i} \rtimes \bar{\pi}_0 \cong \times_{i=1}^r \mathrm{Sp}(\rho_i^\vee, c_i, d_i) \cdot |^{x_i} \rtimes \pi_0^\vee \\ &\cong \left(\times_{i=1}^r \mathrm{Sp}(\rho_i, c_i, d_i) \cdot |^{-x_i} \rtimes \pi_0 \right)^\vee \cong \left(\times_{i=1}^r \mathrm{Sp}(\rho_i^\vee, c_i, d_i) \cdot |^{x_i} \rtimes \pi_0 \right)^\vee \end{aligned}$$

is isomorphic to $\pi^\vee$. Hence, $|I(\rho, c, d)| = |I(\rho^\vee, c, d)|$ and $\{x_i \mid i \in I(\rho, c, d)\} = \{x_i \mid i \in I(\rho^\vee, c, d)\}$ as multisets. Moreover, if $\rho_1 \cong \rho$, $\rho_2 \cong \rho^\vee$, $(c_1, d_1) = (c_2, d_2) = (c, d)$ and $x_1 = x_2$, then since

$$\pi \cong \mathrm{Sp}(\rho, c, d) \cdot |^{x_2} \times \mathrm{Sp}(\rho, c, d) \cdot |^{-x_2} \rtimes \pi'$$

is an irreducible unitary representation with $\pi' = \times_{i=3}^r \mathrm{Sp}(\rho_i, c_i, d_i) \cdot |^{x_i} \rtimes \pi_0$, by (UR), we see that

$$(\mathrm{Sp}(\rho, c, d) \cdot |^{x_2} \times \mathrm{Sp}(\rho, c, d) \cdot |^{-x_2}) \boxtimes \pi'$$

is also unitary. In particular, π' is unitary. Repeating this argument, we may assume that $\rho_i \cong \rho_i^\vee$ for $1 \leq i \leq r$.

Suppose that $\rho \in \mathcal{C}_{\mathrm{sd}}^{\mathrm{GL}}$. If $\rho_1 \cong \rho_2 \cong \rho$ and $(c_1, d_1) = (c_2, d_2) = (c, d)$, then by (CS), we see that

$$\mathrm{Sp}(\rho, c, d) \cdot |^{x_2} \times \mathrm{Sp}(\rho, c, d) \cdot |^{x_2} \rtimes \pi' \cong \mathrm{Sp}(\rho, c, d) \cdot |^{x_2} \times \mathrm{Sp}(\rho, c, d) \cdot |^{-x_2} \rtimes \pi'$$

is an irreducible unitary representation with $\pi' = \times_{i=3}^r \mathrm{Sp}(\rho_i, c_i, d_i) \cdot |^{x_i} \rtimes \pi_0$. By (UR), we see that π' is unitary. Repeating this argument, we may assume that $I(\rho, c, d)$ is at most a singleton for each $\rho \in \mathcal{C}_{\mathrm{sd}}^{\mathrm{GL}}$ and (c, d) .

If $\mathrm{Sp}(\rho_1, c_1, d_1) \rtimes \pi_0$ is irreducible, then by [19, Theorem 1.2], we see that $\mathrm{Sp}(\rho_1, c_1, d_1) \rtimes \pi'$ is also irreducible with $\pi' = \times_{i=2}^r \mathrm{Sp}(\rho_i, c_i, d_i) \cdot |^{x_i} \rtimes \pi_0$. By (CS) and (UR), the unitarity of π implies that π' is also unitary. Repeating this argument, we may assume that $\mathrm{Sp}(\rho_i, c_i, d_i) \rtimes \pi_0$ is reducible for $1 \leq i \leq r$.

We have thus shown that we may assume the following:

- (1) $\pi = \times_{i=1}^r \mathrm{Sp}(\rho_i, c_i, d_i) \cdot |^{x_i} \rtimes \pi_0$ is an irreducible unitary representation;
- (2) $\rho_i \in \mathcal{C}_{\mathrm{sd}}^{\mathrm{GL}}$ and $0 < x_i < \frac{1}{2}$ for $1 \leq i \leq r$;
- (3) for $i \neq j$, we have $\rho_i \not\cong \rho_j$ or $(c_i, d_i) \neq (c_j, d_j)$;
- (4) $\mathrm{Sp}(\rho_i, c_i, d_i) \rtimes \pi_0$ is reducible for $1 \leq i \leq r$.

Remark that the last condition implies that any irreducible summand of the unitary induction $\times_{i=1}^r \mathrm{Sp}(\rho_i, c_i, d_i) \rtimes \pi_0$ is of good parity. The goal is to show that $r = 0$ in this case. For the sake of a contradiction, we assume that $r > 0$.

Proposition 8.4. *Keep the above notation and assumptions. Let $P = MN$ be the maximal parabolic subgroup of G such that $\pi_M = (\times_{i=1}^r \mathrm{Sp}(\rho_i, c_i, d_i)) \boxtimes \pi_0 \in \mathrm{Irr}(M)$. Then there exists an A -parameter $\psi_M \in \Psi(M)$ with $\pi_M \in \Pi_{\psi_M}$ such that the normalized intertwining operator $R_P(w, \pi_M, \psi_M)$ is not a scalar operator, where w is the unique non-trivial Weyl element in $W(M, G)$.*

Proof. By [19, Corollary 3.33], there exist $1 \leq i \leq r$ and an irreducible summand π' of

$$\bigtimes_{\substack{1 \leq j \leq r \\ j \neq i}} \mathrm{Sp}(\rho_j, c_j, d_j) \rtimes \pi_0$$

such that $\mathrm{Sp}(\rho_i, c_i, d_i) \rtimes \pi'$ is reducible. For simplicity, set $\rho = \rho_i$ and $(c, d) = (c_i, d_i)$. Since π' is unitary and of good parity, we can write $\pi' = \pi(\mathcal{S}')$ for some modified extended multisegment $\mathcal{S}'$ as in Definition 8.3 that may be assumed to be standard. Then by [19, Corollary 3.30], there are two irreducible summands π_1 and π_2 of $\mathrm{Sp}(\rho, c, d) \rtimes \pi'$ associated to $\mathcal{S}_1 = \mathcal{S}' \cup \{([C, D]_\rho, \nu), ([C, D]_\rho, \nu)\}$ and $\mathcal{S}_2 = \mathcal{S}' \cup \{([C, D]_\rho, \nu + 2), ([C, D]_\rho, \nu + 2)\}$, respectively, for some $\nu \in \mathbb{Z}$ with $\nu \equiv d \pmod{2}$, where $C = \frac{c+d}{2} - 1$ and $D = \frac{c-d}{2}$. Note that $\mathcal{S}_1$ and $\mathcal{S}_2$ are attached to the same A -parameter $\psi \in \Psi(G)$ as in [19, Definition 3.3(3d)], and that $\pi_1, \pi_2 \in \Pi_\psi$. Moreover, we may assume that ψ comes from an A -parameter $\psi_M \in \Psi(M)$ such that $\pi_M \in \Pi_{\psi_M}$. That is,

$$\psi = \left(\bigoplus_{j=1}^r \rho_j \boxtimes S_{c_j} \boxtimes S_{d_j} \right)^{\oplus 2} \oplus \psi_0$$

with $\pi_0 \in \Pi_{\psi_0}$. By the explicit description of $\mathcal{F}$ in Definition 8.3 (2), together with [5, Theorem 3.6] (corrected in [7, Appendix A]), we see that

$$\prod_{j=1}^r \frac{\langle e(\rho_j, c_j, d_j), \pi_1 \rangle_\psi}{\langle e(\rho_j, c_j, d_j), \pi_2 \rangle_\psi} = (-1)^{\frac{|\nu+2| - |\nu|}{2}} \left((-1)^{d-1} \frac{\mathrm{sgn}(\nu)}{\mathrm{sgn}(\nu+2)} \right)^d.$$

A case-by-case computation shows that the right-hand side is always equal to -1 .

Now by the local intertwining relation proved by Arthur [3, Theorem 2.4.1], the normalized local intertwining operator $R_P(w, \pi_M, \psi_M)$ acts on π_k by the scalar $\prod_{j=1}^r \langle e(\rho_j, c_j, d_j), \pi_k \rangle_\psi$ for $k = 1, 2$. Therefore, the eigenvalue of $R_P(w, \pi_M, \psi_M)$ on π_1 differs from that on π_2 . $\square$

In particular, by (RP1) in Proposition 2.4, for sufficiently small $s > 0$, the irreducible representation

$$\times_{i=1}^r \mathrm{Sp}(\rho_i, c_i, d_i) \cdot |^s \rtimes \pi_0$$

is not unitary. On the other hand, by (CS) and the unitarity of π , it must be unitary. This contradiction proves the “only if” part of Theorem 8.1.

9. ODDS AND ENDS

We have described the unitary dual of split $\mathrm{SO}_{2n+1}(F)$ and $\mathrm{Sp}_{2n}(F)$ in the good parity case, and even slightly beyond it. To go further, several new phenomena arise, which we explain in this section.

9.1. One-parameter complementary series. The first step beyond the good parity case is to consider representations of the form

$$\Pi_s = \mathrm{Sp}(\rho, a, b) | \cdot |^s \rtimes \pi_0,$$

where

- π_0 is of Arthur type and of good parity;
- $\rho \in \mathcal{C}_{\mathrm{sd}}^{\mathrm{GL}}$;
- $s > 0$.

To study this, it is important to consider the *first reducibility point*

$$\mathrm{FRP}(\mathrm{Sp}(\rho, a, b), \pi_0) = \inf \{s \geq 0 \mid \mathrm{Sp}(\rho, a, b) | \cdot |^s \rtimes \pi_0 \text{ is reducible}\}.$$

It is a half-integer and can be computed algorithmically ([6, Theorem 5.3, Corollary 5.4]). One difficulty is that $\mathrm{FRP}(\mathrm{Sp}(\rho, a, b), \pi_0)$ genuinely depends on the pair (a, b) , unlike in the earlier settings treated in [31] and [41].

If $s \equiv \mathrm{FRP}(\mathrm{Sp}(\rho, a, b), \pi_0) \pmod{\mathbb{Z}}$, then all irreducible subquotients of Π_s are of good parity, that we already understand. We therefore need to consider the remaining case.

By (CS), we know that if $s < \mathrm{FRP}(\mathrm{Sp}(\rho, a, b), \pi_0)$, then the irreducible representation

$$\mathrm{Sp}(\rho, a, b) | \cdot |^s \rtimes \pi_0$$

is unitary. However, the converse does not hold, as the following example shows.

Example 9.1. Let $\pi_0 \in \mathcal{C}^G$, and let $\rho \in \mathcal{C}_{\mathrm{sd}}^{\mathrm{GL}}$ be such that $\mathrm{FRP}(\rho, \pi_0) = 3$. (For example, we can take $\rho = \mathbf{1}_{\mathrm{GL}_1(F)}$ and the unique supercuspidal representation $\pi_0 \in \Pi_{\phi_0}$ with $\phi_0 = S_1 \oplus S_3 \oplus S_5 \in \Phi_{\mathrm{temp}}(\mathrm{Sp}_8(F))$.) Set $\pi = \rho \rtimes \pi_0 \in \mathrm{Irr}_{\mathrm{good}}^G$. Then $\mathrm{FRP}(\rho, \pi) = 1$, but by [49],

$$\rho | \cdot |^s \rtimes \pi$$

is irreducible and unitary for all $1 < s < 3$. Indeed, since $\rho | \cdot |^s \rtimes \pi_0$ is unitary by (CS), its unitary induction $\pi = \rho \rtimes (\rho | \cdot |^s \rtimes \pi_0)$ is also unitary.

This example suggests that the following questions are important for understanding the full unitary dual.

Question 9.2. (1) Can one describe in simple terms the set

$$\Pi_{\mathrm{AT}, \mathrm{prime}}$$

consisting of “prime” representations of Arthur type (of good parity), in the sense that they cannot be written as irreducible products

$$\mathrm{Sp}(\rho, a, b) | \cdot |^s \rtimes \pi_0$$

with π_0 of Arthur type (of good parity) and $0 \leq s < \mathrm{FRP}(\mathrm{Sp}(\rho, a, b), \pi_0)$?

(2) For $\pi_0 \in \Pi_{\mathrm{AT}, \mathrm{prime}}$ and $s > 0$ with

$$s \not\equiv \mathrm{FRP}(\mathrm{Sp}(\rho, a, b), \pi_0) \pmod{\mathbb{Z}},$$

if the irreducible representation

$$\mathrm{Sp}(\rho, a, b) | \cdot |^s \rtimes \pi_0$$

is unitary, must it be the case that

$$s < \mathrm{FRP}(\mathrm{Sp}(\rho, a, b), \pi_0) ?$$

The first question can in principle be answered using the results of [6], but the algorithm given there seems insufficient to address the second question. We believe that the set $\Pi_{\text{AT,prime}}$ is the classical analogue of the Speh representations from which one constructs the full unitary dual by parabolic induction.

9.2. Some examples. Finally, we give some explicit examples of unitary representations, beyond the good parity case. In this subsection, we set $\rho = \mathbf{1}_{\text{GL}_1(F)}$, and we drop ρ from the notations.

Let σ be the unique irreducible supercuspidal representation in the A -packet Π_ψ associated to $\psi = S_5 \boxtimes S_5 \in \Psi(\text{Sp}_{24}(F))$. Then $\text{FRP}(\text{Sp}(2, 2), \sigma) = 4$. Fix $0 < \epsilon < \frac{1}{2}$. Since the induced representation

$$\text{Sp}(2, 2)| \cdot |^{-s} \rtimes \sigma$$

is irreducible for $0 \leq s \leq 2 + \epsilon$, we see that $\text{Sp}(2, 2)| \cdot |^{-(2+\epsilon)} \rtimes \sigma$ is unitary. We consider

$$\Pi_s = \text{Sp}(2, 2)| \cdot |^{-(2+\epsilon)} \times \text{Sp}(2, 2)| \cdot |^{-s} \rtimes \sigma.$$

It is irreducible for $0 \leq s < \epsilon$. Therefore, all irreducible subquotients of Π_ϵ are unitary. Note that $\text{Sp}(2, 2)| \cdot |^{-(2+\epsilon)} \times \text{Sp}(2, 2)| \cdot |^{-\epsilon}$ contains $L(\mathbf{m})| \cdot |^{-(1+\epsilon)}$ as an irreducible subquotient, where $\mathbf{m}$ is Leclerc's example (see [33]) given by

$$\mathbf{m} = [-1, -2] + [0, 0] + [1, -1] + [2, 1].$$

Hence

$$L(\mathbf{m})| \cdot |^{-(1+\epsilon)} \rtimes \sigma = L(\Delta[-2 - \epsilon, -3 - \epsilon], | \cdot |^{-1-\epsilon}, \Delta[-\epsilon, -2 - \epsilon], \Delta[\epsilon, \epsilon - 1]; \sigma)$$

is unitary. (The irreducibility of this parabolic induction follows from [32, Lemma 6.2].) However, it is difficult to see from the Langlands data alone why the right-hand side is unitary. This suggests that further notions may be necessary to fully describe the unitary dual of classical groups.

Additional difficulties arise when considering the limit $\epsilon \rightarrow \frac{1}{2}$. Since all irreducible subquotients of the complementary series are unitary, we conclude that

$$\pi = L\left(\Delta\left[-\frac{5}{2}, -\frac{7}{2}\right], | \cdot |^{-\frac{3}{2}}, \Delta\left[-\frac{1}{2}, -\frac{5}{2}\right], \Delta\left[\frac{1}{2}, -\frac{1}{2}\right]; \sigma\right)$$

is also unitary.

We see from the above argument that π is not isolated in the unitary dual. However, we do not know how to prove this fact directly from the Langlands data, since the induced representation

$$\Pi = L\left(\Delta\left[-\frac{5}{2}, -\frac{7}{2}\right], | \cdot |^{-\frac{3}{2}}, \Delta\left[-\frac{1}{2}, -\frac{5}{2}\right], \Delta\left[\frac{1}{2}, -\frac{1}{2}\right]\right) \rtimes \sigma$$

is reducible. (To see that, one can check that Π is preserved under Aubert duality, whereas π is not.)

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DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY, KITASHIRAKAWA-OIWAKE-CHO, SAKYO-KU, KYOTO, 606-8502, JAPAN

Email address: atobe@math.kyoto-u.ac.jp

DEPARTAMENTO DE ÁLGEBRA AND INSTITUTO DE MATEMÁTICAS (IMUS), UNIVERSIDAD DE SEVILLA, C/
TARFIA S/N, 41012 SEVILLA

UNIVERSITY OF VIENNA, FAKULTÄT FÜR MATHEMATIK, OSKAR-MORGENSTERN-PLATZ 1, 1090 WIEN

Email address: `alberto.minguez@univie.ac.at`